INTRODUCTION

DYNAMIC SYSTEM MODEL

\[ \dot{x} = f(t, x(t), u(t)), \quad \forall t \geq 0, \]  
\[ t \in \mathbb{R}^+ \text{ (time), } x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m \]

\[ f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \]

\[ x(t) : \text{state} \]
\[ u(t) : \text{input or control} \]

\[ y = h(t, x, u) \]
\[ y \in \mathbb{R}^q \text{ (output)} \]

\[ \dot{x}_1 = f_1(t, x_1, x_2, \ldots, x_n, u_1, \ldots, u_m) \]
\[ \dot{x}_2 = f_2(t, x_1, x_2, \ldots, x_n, u_1, \ldots, u_m) \]
\[ \vdots \]
\[ \dot{x}_n = f_n(t, x_1, x_2, \ldots, x_n, u_1, \ldots, u_m) \]

\[ x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \quad f(t, x, u) = \begin{bmatrix} f_1(t, x, u) \\ f_2(t, x, u) \\ \vdots \\ f_n(t, x, u) \end{bmatrix} \]

- **Unforced Dynamic System**
  \[ \dot{x} = f(t, x) \quad \text{(independent of } u) \]

- **Forced Dynamic System**
  \[ \dot{x} = f(t, x, u) \]

**Note:** If \( u = g(x) \) function of time yields an unforced system

- If \( f \) independent of \( t \), then system is **autonomous** (time-invariant)
  - otherwise **nonautonomous** (time-varying)

- **Analysis** for a given \( u(t) \) analyze \( x(t) \) behavior
- **Design** find \( u \) to have some specified properties of \( x(t) \).
**EQUILIBRIUM**

\( \mathbf{x}_0 \in \mathbb{R}^n \) is an equilibrium of the unforced system

\[
\dot{x} = f(t, x(t)) \quad \text{if} \quad f(t, \mathbf{x}_0) = 0 \quad \forall t \geq 0
\]

If \( \mathbf{x}_0 \) is an equilibrium, then

\[
\dot{x}(t) = f(t, x(t)) \quad \forall t > t_0 \quad ; \quad x(t_0) = x_0
\]

has the unique solution

\[
x(t) = x_0 \quad ; \quad t > t_0
\]

i.e. if system starts in equilibrium, it should remain in that state thereafter.

\[ \text{unique equilibrium point} \]

\[ \text{infinite equilibrium points} \]

\[ \text{unique 3 equilibrium points} \]

\[
t = 0
\]

\[
\mathbf{x}(0) = \mathbf{x}_0
\]
**Nonlinearity**

A function \( f \) is linear if:

1. \( f(u_1 + u_2) = f(u_1) + f(u_2) \) for any \( u_1 \) and \( u_2 \) in the domain of \( f \)
2. \( f(ku) = k f(u) \) for any \( u \) in the domain of \( f \) and for any real number \( k \).

If \( f \) is not linear, then \( f \) is nonlinear.

**Nonlinearities**

- Inherent (system dynamics)
- Intentional or artificial (introduced by \( u \))
- Continuous
- Discontinuous (hard nonlinearity)

When nonlinearity is continuous, then the behavior of the system can be approximated by a linear system in a small range.

**Linear System (LTI, linear time-invariant system)**

\[ x = Ax \]

- \( A \) is the system matrix

  - A linear system has a unique equilibrium point if \( A \) is nonsingular
  - The equilibrium point is stable if all eigenvalues of \( A \) have negative real parts, regardless of initial conditions.

- The transient response consists of natural modes.
- The general solution can be solved analytically.
- For a forced system:
  \[ \dot{x} = Ax + Bu \]

  - The system satisfies the principle of superposition
  - Sinusoidal input leads to sinusoidal output of the same frequency.
  - We can use Laplace techniques.

If \( \dot{x} = Ax \) is asymptotically stable, then for \( \dot{x} = Ax + Bu \), bounded-input bounded-output (BIBO) stability is implied.
Nonlinear Systems

1) Multiple Equilibrium Points:

Consider example

\[ x = -x + x^2 \]

with initial condition \( x(0) = x_0 \).

Equilibrium points are given by solving

\[ -x + x^2 = 0 \]

\[ x(x-1) = 0 \]

\( x = 0 \) and \( x = 1 \)

By integrating (1) we get

\[ x(t) = \frac{x_0 e^{-t}}{1 - x_0 + x_0 e^{-t}} \]

System trajectory stable or unstable depending on initial conditions.

Study the linearization

\[ \dot{x} = -x, \quad x(0) = x_0 \]

\[ x(t) = x_0 e^{-t} \]

Single equilibrium at 0 stable for any initial value.

2) Failure escape time for nonlinear systems

- Unstable nonlinear systems can have the state go to infinity as time \( \to \) infinity.
- Nonlinear systems can have states go to infinity in finite time (as shown in example above).
3) \( \hat{x} = xu \)
   for \( u = -1 \), system is stable converging to 0
   for \( u = 1 \), " " unstable

4) Look at \( \dot{y} = -w^2 y, \quad y(0) = y_0, \dot{y}(0) = \dot{y}_0 \)
   \( y(t) = A \sin(\omega t) + B \cos(\omega t) \); \( A \) and \( B \) depending on \( y_0 \) and \( \dot{y}_0 \)
   
   \[
   \begin{align*}
   x_1 &= y_0 \\
   x_2 &= \dot{y}_0 
   \end{align*}
   \]
   
   replace \( s^2 + \omega^2 = 0 \) \( \Rightarrow \lambda = \pm j\omega \)

   ![Amplitude of oscillation depends on \( y_0 \) and \( \dot{y}_0 \), i.e. on \( [x_1, x_2]^\top \) \( \Rightarrow \lambda \) in \( x \)

Limit cycle in a nonlinear system

- fixed amplitude and fixed frequency oscillation
- independent of initial conditions.

5) Bifurcation: As parameters of nonlinear dynamic systems are changed, the stability and even the number of equilibrium points can change. The values of these parameters at which the qualitative behavior of the system changes are called **critical or bifurcation values**.

a) Consider undamped Duffing equation

\[
\ddot{x} + \alpha x + x^3 = 0
\]

As \( \alpha \) varies from positive to negative, one equilibrium point at \( x = 0 \) splits into three at \( x = 0, \sqrt{2}, -\sqrt{2} \). : \( x = 0 \) is a critical bifurcation value. (Pitchfork bifurcation)
6) Hopf Bifurcation: Emergence of limit cycles as parameters are changed. A pair of complex eigenvalues conjugate eigenvalues cross from left half plane to right and produce a limit cycle.

\[ \text{limit cycle} \]

6) Chaos: For stable linear systems, small differences in initial conditions can cause only small differences in output. In chaotic nonlinear systems we see a deterministic system producing unpredictability of output.

Quadratic Iterator at \( \alpha = 4 \) shows chaos.
Final-State Diagram

Final-state diagram for the quadratic iterator and parameter $a$ between 1 and 4.
Error Development

The quadratic iterator $x \rightarrow 4x(1-x)$ applied to two initial values differing by $10^{-6}$ (top and center) and the (absolute) difference of the two signals (bottom).
single order for discrete and 1nd and 2nd order, forced
or third order, nonlinear for continuous systems.
Example of 2nd order forced producing chaos is
\[ 3^{\circ} + 0.05 \dot{x} + x^2 = 7.5 \cos t \]

7) **Subharmonic, harmonic or almost periodic oscillation:**
A stable linear system produces a sinusoidal output
of a frequency equal to the frequency of the
sinusoidal input. Nonlinear system output can become
a multiple or submultiple frequency of the input
or almost periodic oscillations (sum of frequencies
not multiply of each other).

8) **Multiple modes of behavior:** As parameters or inputs
are changed (even smoothly), the behavior can
change from one mode to another, such as
limit cycles, subharmonic, etc.

Homework:
Ex. 1.1, 1.2, 1.3, 1.4, 1.5, 1.7

P.7: Use MATLAB to find the state value after 100 steps
for \( x = 1, 2, 3, 4 \). For each value of \( x \), run
the simulation at least 20 times each with different
starting values for \( x(0) \), i.e.,
\[ x(\text{new}) = a \cdot x(k) \cdot (1 - x(k)) \]

P.8: For
\[ \dot{x} = -\text{sign}(x(t)) \quad x(t) \geq 0, x(0) > 0 \]
where \( \text{sign} x = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases} \)
Find the equilibrium points.
SECOND ORDER SYSTEMS

\[ \dot{x}_1 = f_1(x_1, x_2) \]
\[ \dot{x}_2 = f_2(x_1, x_2) \]

- State plane: Two dimensional plane with \( x_1 \) on \( x \)-axis and \( x_2 \) on \( y \)-axis.
- State plane plot or trajectory: If \( x_1(t) \) and \( x_2(t) \) solution, then plot of \( x_2 \) versus \( x_1(t) \) as \( t \) varies over \( \mathbb{R}^+ \).

If \( \dot{x}_1 = x_2 \), then state plane is called phase plane.

and state plane plot or trajectory called phase "or" .

- Smooth function: A function \( f: \mathbb{R}^2 \to \mathbb{R} \) is smooth if \( f(x_1, x_2) \) has continuous partial derivatives of all orders with respect to all possible combinations of \( x_1 \) and \( x_2 \).

- Vector field: A function \( f: \mathbb{R}^2 \to \mathbb{R}^2 \) is called a vector field if both its components are smooth functions. Also called velocity vector field.

- Vector field \( x \in \mathbb{R}^2 \) is an equilibrium of a vector field \( f \) if \( f(x) = 0 \). Also called singular point.

- If \( f(x) \neq 0 \) then the direction of vector field \( f \) at the point \( x \) is defined by

\[ \Theta_f(x) = \text{atan} \left( \frac{f_2(x)}{f_1(x)} \right) \]

(NOTE: \( \text{atan} \left[ a, b \right] \) is a unique number \( \Theta \), such that \( \Theta \in (-\pi/2, \pi/2) \).

\[ \cos \Theta = \frac{a}{\sqrt{a^2 + b^2}} \quad \text{and} \quad \sin \Theta = \frac{b}{\sqrt{a^2 + b^2}} \]
Example: \( \dot{x} + x = 0 \), given \( x(0) = x_0, \dot{x}(0) = 0 \)

**Solution:**

\[ x(t) = x_0 \cos t \]
\[ \dot{x}(t) = -x_0 \sin t \]

Eliminating \( t \), we get
\[ x^2 + \dot{x}^2 = x_0^2 \]

Define \( x_1 = x \)
\[ x_2 = \dot{x} \]

So \( \dot{x}_1 = x_2 \)
\[ x_2 = -x_1 \]

Define \( x_1 + x_2 = x_0 \)

Constructing Phase Portraits

- Computer Generated
- Others: Analytic method
  - Inclusion of delta, epsilon, etc. methods

**Analytic Method**

1) Eliminate time using solutions of \( x_1(t) \)

and \( x_2(t) \) (shown in Example above)

\[ \Rightarrow \]

\[ \frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} \]

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = -x_1 \]

\[ \Rightarrow \]

\[ \frac{dx_2}{dx_1} = \frac{-x_1}{x_2} \]

\[ \Rightarrow \]

\[ z \frac{dx}{dz} + x = 0 \]

Integration yields
\[ x^2 + x^2 = x_0^2 \]
The Method of Integrals

Note:
\[
\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_0)} = \frac{f_1}{f_2} = \alpha \quad \text{(constant \( \alpha \))}
\]

This gives equation:
\[f_2 = \alpha f_1\]

This is an equation for constant \( \alpha \) slope.

Take different values of \( \alpha \), and solve different equations.

Example:
\[
\begin{align*}
\dot{x}_1 &= x_1 \\
\dot{x}_2 &= -x_1
\end{align*}
\]

\[
\frac{dx_2}{dx_1} = \frac{-x_1}{x_2} = \alpha
\]

\[x_1 + \alpha x_2 = 0\]

\[\alpha = 1 \quad \alpha = 0 \quad \alpha = -1 \quad \alpha = \infty\]

(Study example 1.4)

Obtaining Time from Phase Portraits

\[
\frac{dx}{dt} = \dot{x} \Rightarrow \int_{t_0}^{t} \frac{dx}{x} = \int_{t_0}^{t} \dot{x} \, dt
\]

\[t - t_0 = \int_{t_0}^{t} \frac{dx}{x} \]

i.e. if one use \( \frac{1}{x} \) and \( t \) as the coordinates, then the area under the curve gives time elapsed.
Phase Plane Analysis of Linear System

Take \( \dot{x} + ax + bx = 0 \)

We replace

\[ s^2 + ax + b = (s - \lambda_1)(s - \lambda_2) = 0 \]

\[ \lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} \]

\[ \lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2} \]

Solution

\[ x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \quad \text{for} \quad \lambda_1 \neq \lambda_2 \]

\[ x(t) = k_1 e^{\lambda_1 t} + k_2 t e^{\lambda_1 t} \quad \text{for} \quad \lambda_1 = \lambda_2 \]

If \( b \neq 0 \), only one equilibrium point \((x, y) = 0, \dot{x} = 0\)
(a) stable node

(b) unstable node

(c) saddle point

(d) stable focus

(e) unstable focus

(f) center point
Local behavior of non-linear systems

Suppose $A$ has distinct eigenvalues. Consider $A + \Delta A$, elements of $\Delta A$ are arbitrarily small in magnitude.

Eigenvalues of matrices depend continuously on their parameters.

A small change in $\Delta A$ causes a small change in $A$.

Given any positive number $\varepsilon$, there exist a corresponding positive number $\delta$, such that if the magnitude of each element of $\Delta A$ is less than $\delta$, then the eigenvalues of $A + \Delta A$ will lie in a ball centered at eigenvalues of $A$ with a radius $\varepsilon$.

Hyperbolic equilibrium point

$x = 0$ is a hyperbolic equilibrium point of $\dot{x} = Ax$, if $A$ has no eigenvalues with zero real part.

Non-linear systems whose linearization has a hyperbolic eq. point show the linearized local behavior.

(Other ones, one cannot predict)

<table>
<thead>
<tr>
<th>Equilibrium of linearized system</th>
<th>Equilibrium of non-linear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stable node</td>
<td>Stable focus</td>
</tr>
<tr>
<td>Unstable node</td>
<td>Stable focus unstable</td>
</tr>
<tr>
<td>Saddle</td>
<td>Stable focus saddle</td>
</tr>
<tr>
<td>Stable Focus</td>
<td>Unstable focus</td>
</tr>
<tr>
<td>Unstable focus</td>
<td>Center</td>
</tr>
</tbody>
</table>

To determine stability, one must consider the non-linear terms.
Given \( \dot{x} = f(x) \)

Linearization yields (at \( x = 0 \))

Note: Here \( x = 0 \) taken as equilibrium.

If \( x = 0 \) not an equilibrium, define change of coordinates to accomplish that.

\[
\dot{x} = \left( \frac{\partial f}{\partial x} \right)_{x=0} x + \text{h.o.t.}(x)
\]

higher order terms

\[
\therefore A = \left( \frac{\partial f}{\partial x} \right)_{x=0}
\]

for \( \dot{x} = Ax \)

**Limit Cycle**

Isolated closed curve in the phase plane.

Compare it to centers which are not isolated.

Stable: all trajectories converge

Unstable: diverge

Semi-stable: some converge, some diverge
Example

\[ \dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1) \]
\[ \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1) \]

Use \( r = (x_1^2 + x_2^2)^{1/2} \)
\( \theta = \tan^{-1}\left(\frac{x_2}{x_1}\right) \)

\[ \therefore \frac{dr}{dt} = -r(r^2 - 1) \]
\[ \frac{d\theta}{dt} = -1 \]

Notice at \( r = 1, \dot{r} = 1 \)
for \( r < 1, \dot{r} > 0 \)
for \( r > 1, \dot{r} < 0 \)

\[ \therefore \text{stable limit cycle for } r = 1. \]

(Also see analytical solution)
\[ r(t) = \frac{1}{(1 + c_0 e^{-2t})^{1/2}} \]
\[ \theta(t) = \theta_0 - t \]
\[ c_0 = \frac{1}{r_0^2} - 1 \]

\[ \phi(t) = f(x) \]
the semi orbit through \( y \Rightarrow r^+(y) = \{\phi(t, y) | 0 \leq t < \infty\} \)
\( \phi \) is solution of \( \dot{x} = f(x) \) stable \( \phi(y) = y \)

the semi orbit through \( y \Rightarrow r^-(y) = \{\phi(t, y) | -\infty < t \leq 0\} \)
\( r \) is a true limit point of solution \( \phi(t, y) \) (for \( t \geq 0 \)) if there is a sequence of \( t_n, y \) with \( t_n \to x \) as \( n \to \infty \) s.t.
\[ \phi(t_n, y) \to r \] as \( n \to \infty \)
The set of all the limit points of φ(τ,y) is L⁺, the true limit set of φ(τ,y).

φ(τ,y) is bounded from above, the limit set is non-empty, invariant, compact, and φ(τ,y) approaches its positive limit set as τ → +∞ (see (B.1))

(Poincare-Bendixson)

Let γ⁺ be a bounded true semi-orbit of x = f(x) and L⁺ be its true limit set.

If L⁺ contains no equilibrium point, then it is a periodic orbit.

(C.10)

(Bendixson Criterion)

Let D be a simply connected region in \( \mathbb{R}^2 \).

The expression \( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \) is not identically zero and does not change sign, then x = f(x) has no periodic orbits lying entirely in D.

For a limit cycle, using Stokes's theorem.

\[ \int (f_2 \, dx_1 - f_1 \, dx_2) = 0 \Rightarrow \int \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) \, dx_1 \, dx_2 = 0 \]

(Poincare)

If a limit cycle exists in a second order system then \( N = \# \text{o.c. nodes, centers and foci enclosed by a limit cycle and s is the} \)

\# of enclosed saddle points.
Oscillation when
\[ x(t+T) = x(t) \]
with a non-trivial solution
\[ c \neq x(t) \neq 0 \]

For linear systems
when \( \lambda = \pm j \beta \), we see oscillation (damped or undamped)

1. Unstable (any perturbation can destabilize it)
2. Amplitude dependent on initial conditions.

Nonlinear oscillations (limit cycles)

1. Stable
2. Amplitude independent of initial conditions

Numerical Construction of Phase Portraits

1. Select areas around equilibrium points
2. Choose initial points
3. Take each initial point \( x_0 \)
   and solve for trajectories in forward and backward time by
   \[ x = f(x), \quad x(0) = x_0 \]
   \[ x = -f(x), \quad x(0) = x_0 \]

Homework:
- 2.1, 1.15, 1.16, 1.17, 1.18
- Analyze limit cycle for
  \[ x_1 = x_2 + x_1(x_1^2 + x_2^2 - 1) \]
  \[ x_2 = -x_1 + x_2(x_1^2 + x_2^2 - 1) \]