**Example:**

\[ \dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2) \]
\[ \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2) \]

\((0, 0)\) equilibrium

Take \( V(x) = x_1^2 + x_2^2 \)

\( \dot{V}(x) = \sum_{i=1}^{2} \frac{\partial V}{\partial x_i} f(x_i) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = -2(x_1^2 + x_2^2) < 0 \quad \forall (x, y) \neq (0, 0) \)

\( \therefore \) origin globally asymptotically stable.

\( \therefore \) origin is a unique equilibrium point.

**Example**

\[ \ddot{\theta} + \dot{\theta} + \sin \theta = 0 \]

\( V(x) = (1 - \cos \theta) + \theta^2 \Rightarrow \dot{V}(x) = \ddot{\theta} \sin \theta + \dot{\theta} \dot{\theta} = -\dot{\theta}^2 \leq 0 \) (stable)

Take \( V(x) = 2(1 - \cos \theta) + \frac{1}{2} \dot{\theta}^2 + \frac{1}{2} (\dot{\theta} + 0)^2 \Rightarrow \dot{V}(x) = -(\dot{\theta}^2 + \theta \sin \theta) \leq 0 \)

Then second \( V(x) \) shows local asymptotic stability also.

- In general many sets Lyapunov functions possible

If \( V \) is a Lyapunov function, then so is \( V_1 = \alpha V \) for any \( \alpha \) positive constant \( p \) and any scale \( \alpha \).

- Lyapunov theorem is a sufficient condition, if one \( V \) doesn't work, look for another.

**Instability Theorem** (Chetaev's Theorem)

Let \( x_0 \) be equilibrium point for \( \dot{x} = f(x) \), let \( V : D \rightarrow \mathbb{R} \)

be a continuously differentiable function such that \( V(0) = 0 \)

and \( V(x_0) > 0 \) for some \( x_0 \) with arbitrarily small \( ||x_0|| \).

Define \( U \) as \( U = \{ x \in \mathbb{B}_r \ | V(x) > 0 \} \), then if \( \dot{V}(x) > 0 \)

\( \forall U \Rightarrow x = 0 \) is unstable.

E.g. \( V(x) = \frac{1}{2} (x_1^2 + x_2^2) \) is true in neighborhood around the line \( x_2 = 0 \)

\[ \mathbb{B}_r \text{ neighborhood for positive } V(x) \]
**Invariant Set:** A set $G$ is an invariant set for a dynamic system if every system trajectory starting in $G$ remains in $G$ for all future time.

**La Salle's Theorem**

**Local Invariant Set Theorem:** Consider $\dot{x} = f(x)$ with $f$ continuous and let $V(x)$ be a scalar function with continuous first partial derivatives.

- For some $l > 0$, the region $R_l$ defined by $V(x) < l$
- $\dot{V}(x) \leq 0 \quad \forall x \in R_l$

Let $R$ be the set of all points within $R_l$ where $V(x) = 0$, and $M$ be the largest invariant set in $R$. Then every solution $x(t)$ originating in $R$ tend to $M$ as $t \to \infty$.

To prove this use Barbalat's Lemma: If the differentiable function $f(t)$ has a finite limit as $t \to \infty$, and if $f$ is uniformly continuous, then $f \to 0$ as $t \to \infty$.

If a differentiable function $f$ has a derivative which exists and is bounded then $f$ is uniformly continuous.

**Application to Lyapunov:** For $\dot{x} = f(x)$, $V(x)$ a scalar function with continuous partial derivatives. If in a neighborhood $R$ of origin

- $V(x)$ is locally $+ve$ definite
- $\dot{V}$ is $-ve$ semi definite

The set $R$ defined by $V(x) = 0$ contains no trajectory of $x = f(x)$ other than $x \equiv 0$, then $0$ is asymptotically stable.

**Global Version:** $\dot{x} = f(x)$, $V(x)$ scalar with continuous partial derivatives. If

- $\dot{V}(x) \leq 0$ are all $R^n$
- $V(x) \to 0$ as $\|x(t)\| \to \infty$
- $V(x) \to \infty$ as $\|x(t)\| \to \infty$

Let $R$ be the set of all points where $\dot{V}(x) = 0$ and $M$ the largest invariant set in $R$ then all solution globally asymptotically converge to $M$ as $t \to \infty$. 

Example: (Domain of Attraction)

\[ \begin{align*}
\dot{x}_1 &= x_1 (x_1^2 + x_2^2 - 1) - x_2 \\
\dot{x}_2 &= x_1 + x_2 (x_1^2 + x_2^2 - 1)
\end{align*} \]

\[ V(x_1, x_2) = x_1^2 + x_2^2 \]

\[ \dot{V}(x) = 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1) < 0 \quad \text{in} \quad \{x_1 x_2^2 < 1\} \]

\[ \Rightarrow \text{domain of attraction} \quad V(x) < 1, \quad \text{i.e.} \quad x_1^2 + x_2^2 < 1 \]

Example: Attractive Limit Cycle

\[ \begin{align*}
\dot{x}_1 &= x_2 - x_1 [x_1^2 + 2x_2^2 - 10] \\
\dot{x}_2 &= -x_1^3 - 3x_1^2 x_2 + 2x_2^3 - 10
\end{align*} \]

Set \[ x_1^2 + 2x_2^2 = 10 \] is invariant, \[ \Rightarrow \]

\[ \frac{\text{d}r}{\text{d}t} = (x_1^2 + 2x_2^2 - 10) = -y(x_1^2 + 2x_2^2)(x_1^2 + 2x_2^2 - 10) \]

The invariant set is given by \[ x_1 = x_2 \Rightarrow \text{a limit cycle} \]

\[ \begin{align*}
V &= (x_1^2 + 2x_2^2 - 10)^2 \\
\dot{V} &= -8(x_1^2 + 2x_2^2)(x_1^2 + 2x_2^2 - 10)^2 \\
\dot{V} &\leq 0 \quad \text{and} \quad V = 0 \Rightarrow x_1^2 + 2x_2^2 = 10 \rightarrow \text{Limit cycle} \]

You can take \[ x_1 V(x) < 100 \] which is all interior limit cycle except \( (0,0) \) which is stable limit cycle.
Techniques for Guessing Lyapunov Functions.

1. Physical Attributes (e.g., energy)

\[\mathbf{x}(\dot{q}) = \dot{\mathbf{x}} + \mathbf{g}(\mathbf{q}) + \mathbf{f}(\mathbf{q}) + \mathbf{g}_c(\mathbf{q}) = \mathbf{L} - \mathbf{L} \quad \text{torque}\]

\[
\mathbf{L} = -k_d \dot{\mathbf{q}} - k_p \mathbf{q} + \mathbf{g}(\mathbf{q})
\]

\[
\mathbf{V} = \frac{1}{2} \left[ \dot{\mathbf{q}}^T \mathbf{K} \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{K} \dot{\mathbf{q}} \right]
\]

\[
\dot{\mathbf{V}} = \dot{\mathbf{q}}^T (\dot{\mathbf{L}} - \mathbf{g}) + \dot{\mathbf{L}}^T \mathbf{K} \dot{\mathbf{q}} \quad \text{using (1) gives}
\]

\[
\dot{\mathbf{V}} = -\dot{\mathbf{q}}^T k_d \dot{\mathbf{q}} \leq 0
\]

\[
\dot{\mathbf{V}} = 0 \Rightarrow \dot{\mathbf{q}} = 0 \quad \text{(but that will have non zero acceleration)}
\]

\[
\therefore (0,0) \quad \text{globally asymptotically stable}
\]

2. Kravovski's Method: For \( \dot{x} = f(x) \), equilib at \( x = 0 \). Let

\[
A(x) = \frac{\partial f}{\partial x}
\]

If \( f = A + A^T \) is -ve def. in a neighborhood \( \mathcal{N} \), then \( x = 0 \)

is asymptotically stable. If \( \mathcal{N} \) is \( \mathbb{R}^n \) and \( V(x) \to \infty \)

as \( \| x \| \to \infty \), then \( x = 0 \) is globally asymptotically stable.

\[\text{HINT: Use } V(x) = f^T(x) f(x)\]

3. Generalized Kravovski's Method: \( \dot{x} = f(x) \), equilib \( x = 0 \), \( A(x) = \frac{\partial f}{\partial x} \).

If \( \mathcal{N} \) is symmetric +ve def. matrix and \( \mathcal{N} \), s.t. \( \forall x \leq 0 \)

\[f(x) = A^T y + A y + 0 \]

is -ve semidef. in some neighborhood \( \mathcal{N} \) of the origin, then \( x = 0 \) is asympt. stable.

If \( \mathcal{N} = \mathbb{R}^n \) and \( V(x) \to \infty \) as \( \| x \| \to \infty \), then globally asympt. stable.

\[\text{HINT: Use } V(x) = f^T P f\]
(MATRICES).

**Symmetric square matrix** satisfies: \( M = MT \)

**Skew-symmetric** satisfies: \( M = -MT \)

**Note:** For a skew-symmetric matrix \( x^T M x = -x^T M x \)

\[ x^T M x = 0 \quad \forall x \]

Any matrix (square) \( M = \frac{M + M^T}{2} + \frac{M - M^T}{2} \)

\( \downarrow \text{symmetric} \quad \downarrow \text{skew-symmetric} \)

**Positive definite matrix**

A square \( n \times n \) matrix \( M \) is **+ve def.**

\[ \text{if} \quad x \neq 0 \quad \Rightarrow \quad x^T M x > 0 \]

(necessary (not sufficient) condition is that all diagonal
elements of \( M \) be +ve (\( \forall i \geq 0 \))
also \( M \) is +ve def. iff its principal minors (\( \text{i.e. } M_{11}, M_{11}M_{22} - M_{12}M_{12}, \ldots, \det M \) are all \( > 0 \))
\( \lambda_{\min}(M) \|x\|^2 \leq x^T M x \leq \lambda_{\max}(M) \|x\|^2 \quad \forall x \in \mathbb{R}^n \), \( \lambda(M) \) = eigenvalue of \( M \).

**Example of Krasovskii’s method.**

\[
\begin{align*}
\dot{x}_1 &= -3x_1 + x_2 \\
\dot{x}_2 &= x_1 - x_2 - x_3^2
\end{align*}
\]

\[ A = \frac{\partial f}{\partial x} = \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix} \quad F = A + AT = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \]

Matrix \( F \) is -ve def. (\( \therefore -F \) +ve def.)

\( \therefore \) asympt. stable \( 0 \)

\[ V(x) = f^T(x) f(x) = (-3x_1 + x_2)^2 + (x_1 - x_2 - x_3^2)^2 \]

\( \therefore V(x) \to 0 \) as \( \|x\| \to 0 \) \( 0 \) is globally asympt. stable.

**Variable gradient method**

98. \( V(x) \) is a Lyapunov function for \( \dot{x} = f(x) \), then

\[ \dot{V} = V \nabla x \]

Integrating from 0 to \( t \)

\[ V(x) = \int_0^x \nabla x \ dx \]

Since left-hand side dependent only on \( x \), \( g. a. s. 
integral independent of path.

\[ \therefore \frac{\partial V_{i}}{\partial x_{i}} = \sum_{j=1}^{n} \frac{\partial V_{i}}{\partial x_{j}} \quad \text{(i, j = 1, 2, \ldots, n)} \]
Steps for gradient method

- Take $\nabla V$ as $\nabla V_i = \sum_{j=1}^{n} a_{ij} \dot{x}_j$

- Take $a_{ij}$ s.t. $V$ is +ve def. at least locally

- solve for $a_{ij}$ s.t. $\frac{\partial \nabla V_i}{\partial x_i} = \frac{\partial \nabla V_j}{\partial x_i}$ (for $i,j = 1,2,\ldots,n$)

- Compute $V$ from $\nabla V$ by integration

\[
V(x) = \int_{x_0}^{x} \nabla V dx
\]

one convenient method is

\[
V(x) = \int_{x_0}^{x_1} \nabla V_1(x_1,0,\ldots,0) dx_1 + \int_{0}^{x_2} \nabla V_2(x_1,x_2,0,\ldots,0) dx_2 + \ldots + \int_{0}^{x_n} \nabla V_n(x_1,x_2,\ldots,x_{n-1}) dx_n
\]

- check if $V$ is +ve def.

Example:

\[
\dot{x}_1 = -x_1 \\
\dot{x}_2 = -x_2 + x_1 x_2
\]

assume \ $\nabla V_1 = a_{11} x_1 + a_{12} x_2$

\[\nabla V_2 = a_{21} x_1 + a_{22} x_2\]

curl condition

\[\frac{\partial \nabla V_2}{\partial x_1} = \frac{\partial \nabla V_1}{\partial x_2} \Rightarrow a_{12} + x_2 \frac{\partial a_{22}}{\partial x_2} = a_{21} + \frac{\partial a_{11}}{\partial x_1}\]

choose $a_{11} = a_{22} = 1$, $a_{12} = a_{21} = 0$

\[\nabla V_1 = x_1, \ \nabla V_2 = x_2\]

\[\dot{V} = \nabla V \cdot \dot{x} = -x_1^2 - x_2^2 (1-x_1 x_2)\]

+ve if $1-x_1 x_2 > 0 \Rightarrow \dot{V} \dot{y} - ve .

\[V(x) = \int_{x_0}^{x_1} x_1 dx_1 + \int_{0}^{x_2} x_2 dx_2 = \frac{x_1^2 + x_2^2}{2}\]

For Linear Systems

\[\dot{x} = Ax\]

\[V = x^T P x, \ P \ is \ symmetric \ +ve \ def. \ matrix\]

\[\dot{V} = x^T P \dot{x} + x^T \dot{P} x = x^T (AP + PA)x = -x^T Q x\]

where $AP + PA = -Q$

\[\Rightarrow Q +ve \ def. \ also \Rightarrow \ global \ asymp. \ stable\]
MORE ON LINEARIZATION

\[ \dot{x} = f(x), \quad x_0 \in \mathbb{R}^n, \quad f : \mathbb{R}^n \rightarrow \mathbb{R}^n \]

continuously differentiable function

using mean value theorem,

\[ f(x) = f(0) + \frac{df}{dx}(z)x, \quad \text{where} \quad z \in [0, x] \]

\[ f(0) = 0 \]

\[ \therefore f(x) = \frac{df}{dx}(z)x = \frac{df}{dx}(0) + \left[ \frac{df}{dx}(z) - \frac{df}{dx}(0) \right]x \]

\[ = Ax + g(x) \]

\[ A = \frac{df}{dx}(0) \quad \text{and} \quad g(z) = \left[ \frac{df}{dx}(z) - \frac{df}{dx}(0) \right]x \]

\[ \|g(x)\| \leq \|\frac{df}{dx}(z) - \frac{df}{dx}(0)\| \|x\| \]

\[ \therefore \frac{df}{dx} \text{ is continuous} \quad \|\frac{df}{dx}(z) - \frac{df}{dx}(0)\| \rightarrow 0 \]

as \( \|x\| \rightarrow 0 \)

\[ \frac{\|g(x)\|}{\|x\|} \rightarrow 0 \quad \text{as} \quad \|x\| \rightarrow 0 \]

\[ \therefore \dot{x} = Ax \quad \text{near} \quad x = 0 \]
NONAUTONOMOUS SYSTEMS

**EQUILIBRIUM POINTS**: \( \dot{x} = f(t, x) \), if \( f(t, x^*) = 0 \) \( \forall t \geq t_0 \).

**STABILITY**: Equilibrium point 0 for \( \dot{x} = f(t, x) \) is stable at \( t_0 \) if for any \( R > 0 \), \( \exists \alpha > 0 \) such that
\[
||x(t)|| < \alpha \Rightarrow ||x(t)|| < R \quad \forall t \geq t_0.
\]
Otherwise, the equilibrium point is unstable.

**ASYMPTOTICALLY STABLE**: 0 is asymptotically stable at time \( t_0 \) if

1) it is stable and
2) \( \exists \alpha > 0 \), \( \forall t \geq t_0 \):
\[
||x(t)|| < \alpha \Rightarrow ||x(t)|| \to 0 \quad \forall t \to \infty
\]

**EXPONENTIALLY STABLE**: if \( \exists \alpha > 0 \) for sufficiently small \( x(t_0) \),
\[
||x(t)|| \leq \alpha ||x(t_0)|| e^{-\gamma (t - t_0)} \quad \forall t \geq t_0
\]
then 0 is exp. stable.

**GLOBALLY ASYMPTOTICALLY STABLE**: if \( x(t) \to 0 \quad \forall t \to \infty \)

**Example**: \( \dot{x}(t) = -a(t)x(t) \)

solution \( x(t) = x(t_0) e^{-\int_{t_0}^{t} a(u) du} \)

*stable* if \( a(t) > 0, \forall t > t_0 \)

*asymptotically stable* if \( \int_{t_0}^{\infty} a(u) du = \infty \)

*exponentially stable* if \( \exists T > 0, \forall t > T \) and \( \forall t > T \), \( a(t) \geq \gamma \)

**UNIFORMITY** (Desire systems to be equally stable for all \( t \)).

They should not get less stable at \( t \) necessarily.

**LOCAL UNIFORM STABILITY**: if for any \( R > 0 \), \( \exists \alpha > 0 \), \( R = \alpha(R), \forall t > t_0 \)

\[
||x(t)|| < \alpha \Rightarrow ||x(t)|| < R, \forall t \geq t_0.
\]

(\( R \) independent of \( t_0 \))

**UNIFORM ASYMPTOTIC STABILITY**: if

*uniformly state

\[ \exists B(0) \text{ (ordered } R_0 \text{ independent of } t_0) \), \( \forall t \) if \( x(t_0) \in B(0) \),
then \( x(t) \to 0 \) uniformly in \( t_0 \)

**Uniform Convergence**: if \( R_2 < R_1 \leq R_0 \), \( \exists T(R_1, R_2) > 0, \forall t \geq T(R_1, R_2) \)

\[
||x(t)|| < R_1 \Rightarrow ||x(t)|| < R_2, \forall t \geq T(R_1, R_2)
\]
uniform asymptotic stable \implies asymptotically stable \\
\text{(not necessarily \leq)}

\text{Example: } \dot{x} = \frac{-x}{1+t} \implies x(t) = \frac{1+\frac{x(0)}{1+t}}{1+t}

\text{asymptotic but not \underline{uniform} to zero (\text{\because larger to}}
\text{requires larger } t \text{ to go to zero).}

\text{Exponential stability} \implies \text{uniform asymptotic stability}

\text{Globally uniformly asymptotic stability: } B_{B_0} = \mathbb{R}^n

\text{Locally positive definite } V(t,x) \text{ if }

V(t,0) = 0 \text{ and } t > t_0, V(t,x) \geq V_0(x)

\text{where } V_0(x) \text{ is pd.}

\text{Also globally pd, negative def., semi def., etc.}

\text{Decrescent function if } V(t,0) = 0 \text{ and } V_0(x) \text{ is pd and}

\forall x > 0, V(t,x) \leq V_0(x)

\text{Example: } V(t,x) = (1+t)(x_1^2 + x_2^2)

\therefore x_1^2 + x_2^2 \leq V(t,x) \leq 2(x_1^2 + x_2^2)

\therefore \text{ pd and also decrescent.}

\text{Lyapunov theorem for non autonomous systems}

\text{Stability: If in a ball } B_{B_0} \text{ around equilibrium point } 0,
\text{\exists } V(t,x) \text{ with continuous partial derivative, s.t.}
1) V \text{ is pd}
2) \dot{V} \text{ is nsd}
\therefore 0 \text{ is stable}

\text{Uniform stability: 3) } V \text{ is decrescent}

(V,1) \text{ and (3) \implies uniformly stable } 0

\text{Uniform asymptotic stability (1), (3) and (2) replaced by}

(V,2) \text{ is nsd} \implies \text{uniformly asymptotic stability}

\text{Global uniform asymptotic stability } B_{B_0} = \mathbb{R}^n, V(t,x) \text{ radially}
\text{unbounded, (1), enhanced (2), (3) and (4) also,}
Example
\[ \dot{x}(t) = -x(t) - e^{2t} x_2(t) \]
\[ \dot{x}_2(t) = x_1(t) - x_2(t) \]

\[ V(x(t)) = x_1^2 + (1 + e^{2t}) x_2^2 \]

\[ V(x(t)) \leq -2 \{ x_1^2 - x_1 x_2 + x_2^2 \} + 2 e^{2t} \]

\[ \dot{V}(x(t)) \leq -(x_1 - x_2)^2 - x_2^2, \text{ for } V \text{ nd} \]

\[ \text{V is decreasing, } x(t) \rightarrow \text{ globally uniformly asymptotically stable} \]

**NOTE:** In non-autonomous systems, \( V \) is pd and \( V \) is nd \( \Rightarrow \) guaranteed asymptotic stability.

**Class K Function:** continuous function \( \alpha : R^+ \rightarrow R^+ \) is class \( K \) if

\[ \alpha(0) = 0, \alpha'(p) \geq 0, \forall p > 0, (5) \alpha \text{ is non-decreasing} \]

**Lemma:** \( V(x(t)) \) is locally (or globally) pd, iff \( \exists \alpha(x) \) of class K, s.t. \( V(t, 0) = 0 \) and \( V(x(t)) \geq \alpha(\|x(t)\|) \)

\[ \forall t \geq 0 \text{ and } x \in BR_0 (or R^n) \]

\[ \text{if } V(t, 0) = 0 \text{ and } V(x(t)) \leq \beta(\|x(t)\|), \forall t \geq 0 \text{ and } x \in BR_0 \]

\[ \text{HINT:} \text{ if } V(t, x) \text{ is pd, then } V(x(t)) \geq V_0(x) \]

\[ \text{Take } \alpha(p) = \sup_{\|x\| \leq p} V_0(x) \]

\[ \text{if } V(x(t)) \text{ is decreasing, then } V(t, x(t)) \leq V_0(x) \]

\[ \text{Take } \beta(p) = \sup_{\|x\| \leq p} V_0(x) \]

**Theorem:** If in a neighborhood of \( 0 \leq \|x(t)\| \leq p \) with continuous first order derivative and a class K function \( \alpha \), s.t., \( x \neq 0 \)

\[ \forall x, \dot{x} = 0 \] \[ V(x(t)) \geq \alpha(\|x(t)\|) > 0 \]

\[ \text{then } 0 \text{ is stable.} \]

\[ \forall x \in R^n \] \[ V(x(t)) \leq \beta(\|x(t)\|) = 0 \text{ uniformly stable.} \]

\[ \text{If (1, 3) and 2a replaced by } 2a \text{, } V(x(t)) \leq -V(\|x(t)\|) < 0 \]

\[ \text{if } \text{a K function, then } 0 \text{ uniformly asymptotically stable.} \]

\[ \forall x(0) \text{ and (3) satisfied in } R^n \text{ and } \forall x(\|x(t)\|) \rightarrow 0 \]

\[ \text{as } x \rightarrow 0 \]

96 (1, 2, 3) satisfied in } R^n \text{ and } \forall x(\|x(t)\|) \rightarrow 0 \]

\[ \text{then } 0 \text{ is globally uniformly asymptotically stable.} \]
Proof: We derive the three parts of the theorem in sequence.

Lyapunov stability: To establish Lyapunov stability, we must show that given $R > 0$, there exists $r > 0$ such that (4.6) is satisfied. Because of conditions 1 and 2a,

$$\alpha(||x(t)||) \leq V[x(t), t] \leq V[x(t_o), t_o] \quad \forall t \geq t_o$$

(4.13)

Because $V$ is continuous in terms of $x$ and $V(\theta, t_o) = 0$, we can find $r$ such that

$$||x(t_o)|| < r \quad \Rightarrow \quad V(x(t_o), t_o) < \alpha(R)$$

This means that if $||x(t_o)|| < r$, then $\alpha(||x(t)||) < \alpha(R)$, and, accordingly, $||x(t)|| < R$, $\forall t \geq t_o$.

Uniform stability and uniform asymptotic stability: From conditions 1 and 3,

$$\alpha(||x(t)||) \leq V(x(t), t) \leq \beta(||x(t)||)$$

For any $R > 0$, there exists $r(R) > 0$ such that $\beta(r) < \alpha(R)$ (Figure 4.1). Let the initial condition $x(t_o)$ be chosen such that $||x(t_o)|| < r$. Then

$$\alpha(R) > \beta(r) \geq V[x(t_o), t_o] \geq V[x(t), t] \geq \alpha(||x(t)||)$$

This implies that

$$\forall t \geq t_o, \quad ||x(t)|| < R$$

Uniform stability is asserted because $r$ is independent of $t_o$.

In establishing uniform asymptotic stability, the basic idea is that if $x$ does not converge to the origin, then it can be shown that there is a positive number $a$ such that $-\dot{V}[x(t), t] \geq a > 0$. This implies that

$$V[x(t), t] - V[x(t_o), t_o] = \int_{t_o}^{t} \dot{V} \, dt \leq -(t - t_o) a$$
and thus, that

$$0 \leq V[x(t), t] \leq V[x(t_o), t_o] - (t - t_o) \alpha$$

which leads to a contradiction for large $t$. Let us now detail the proof.

Let $\|x(t_o)\| \leq r$, with $r$ obtained as before. Let $\mu$ be any positive constant such that $0 < \mu < \|x(t_o)\|$. We can find another positive constant $\delta(\mu)$ such that $\beta(\delta) < \alpha(\mu)$. Define $\varepsilon = \gamma(\delta)$ and set

$$T = T(\mu, r) = \frac{\beta}{\varepsilon}$$

Then, if $\|x(t)\| > \mu$ for all $t$ in the period $t_o \leq t \leq t_1 = t_o + T$, we have

$$0 < \alpha(\mu) \leq V[x(t_1), t_1] \leq V[x(t_o, t_o)] - \int_{t_o}^{t_o} \gamma(\|x(s)\|) \, ds \leq V[x(t_o), t_o] - \int_{t_o}^{t_o} \gamma(\delta) \, ds$$

$$\leq V[x(t_o), t_o] - (t_1 - t_o) \varepsilon \leq \beta(r) - T \varepsilon = 0$$

This is a contradiction, and so there must exist $t_2 \in [t_o, t_1]$ such that $\|x(t_2)\| \leq \delta$. Thus, for all $t \geq t_2$,

$$\alpha(\|x(t)\|) \leq V[x(t), t] \leq V[x(t_2), t_2] \leq \beta(\delta) < \alpha(\mu)$$

As a result,

$$\|x(t)\| < \mu \quad \forall t \geq t_o + T \geq t_2$$

which shows uniform asymptotic stability.

**Global uniform asymptotic stability:** Since $\alpha(\cdot)$ is radially unbounded, $R$ can be found such that $\beta(r) < \alpha(R)$ for any $r$. In addition, $r$ can be made arbitrarily large. Hence, the origin $x = 0$ is globally uniformly asymptotically stable.
Barbalat’s Lemma

- $f \to 0 \Rightarrow \hat{f} \to 0$ converges. Example $f(t) = \frac{\cos(\log t)}{t} \to 0$ as $t \to \infty$
- If $f$ converges $\Rightarrow \hat{f} \to 0$. Example $f(t) = e^{-t} \sin(e^{2t})$
- If $f$ is lower bounded and decreasing ($f' \leq 0$) $\Rightarrow f \to f_0$ (limit lemma): If the differentiable function $f(t)$ has a finite limit as $t \to \infty$ and if $\hat{f}$ is uniformly continuous, then $\hat{f}(t) \to 0$ as $t \to \infty$

**EXAMPLE:** Closed loop error dynamics of an adaptive control system for a first order plant with one unknown parameter are

$$\dot{e} = -e + \theta w(t)$$

$$\dot{\theta} = -e w(t)$$

$$V = e^2 + \theta^2$$

$$\dot{V} = 2e(-e + \theta w) + 2\theta(-e w(t)) = -2e^2 \leq 0$$

$\Rightarrow V(t) \leq V(0)$

- $e$ and $\theta$ are bounded
- $\dot{V} = -2e(-e + \theta w) \Rightarrow \dot{V}$ is bounded
- $\dot{V} \to 0 \Rightarrow e \to 0$ as $t \to \infty$