Complementarity Problem:

A pair of variables \((u, y)\) is subject to a complementarity condition if

\[
u \geq 0, \quad y \geq 0, \quad u^Ty = 0
\]

Notation: \(0 \leq y \perp u \geq 0\) (\(\therefore u \geq 0, y \geq 0, u^Ty = 0\))

Example:

\[
\begin{align*}
\begin{array}{c}
\downarrow \\
\rightarrow
\end{array} & \begin{array}{c}
\uparrow \\
\leftarrow
\end{array} \\
\text{1} & \text{2}
\end{align*}
\]

\[
0 \leq y \perp u \geq 0 \quad \text{(1) } i = 0, u \geq 0 \\
\text{2} & \quad \text{0} \leq y \perp u \geq 0 \quad \text{(2) } v = 0, i \geq 0
\]

Linear Complementarity Problem (LCP): (from mathematical programming - optimization and D.R. area)

\(\text{LCP}(q, M)\): given an \(n\)-vector \(q\) and an \(m \times m\) matrix \(M\), find an \(m\)-vector \(z\), s.t.

\[
\begin{align*}
& z \geq 0 \\
& q + Mz \geq 0 \\
& z^T(q + Mz) = 0
\end{align*}
\]

\(z^T(q + Mz) = 0 \quad \text{if and only if } w = q + Mz \quad \text{or } w - Mz = q
\]

\(z^Tq = 0 \quad \text{and } z^TMz = 0
\]

LCP as an optimization problem (Bilinear programming)

\[
\min \sum [y_jw_j + (1-y_j)z_j]
\]

\[
\begin{align*}
& w - Mz = q \\
& w^Tz \geq 0 \\
& z \in \mathbb{R}^n
\end{align*}
\]

\(z\) is complementary.

a) The optimal solution is zero if the solution is feasible.

b) Variable \(y_j\) indicates if \(w_j\) or \(z_j\) is nonzero.

(Note: if \(q \geq 0\), we have an immediate solution, \(w = q, z = 0\))

LCP also a solve of a quadratic minimization problem.

Piecemeal devices circuit:

\[
V_a = \max \left( \frac{1}{2} I_r I_r, I_r \right) = \frac{I_r}{2} I_r, I_r < 0
\]

\[
V_a = \frac{1}{2} I_r u + u
\]

\[
y = -I_r + 2u
\]

\[
0 \leq y \perp u \geq 0
\]

\(\therefore u = \max \left( \frac{1}{2} I_r, 0 \right)\) and \(V_a = \max \left( \frac{1}{2} I_r, I_r \right)\)
Take \( C = 1 \text{F}, L = 1 \text{H}, R = 1 \Omega \); \( x_1 \) = voltage across \( C \); \( x_2 \) = current through \( L \)

\[
\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -1 & -\frac{3}{2} \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ -1 \end{pmatrix} u(t)
\]

\[
y(t) = \begin{pmatrix} 0 & -1 \end{pmatrix} x(t) + 2 u(t)
\]

Graph showing solution for \( x(t) \) and \( y(t) \) with \( 0 \leq y(t) \perp u(t) \geq 0 \)

**Example of well-posed circuit:**

\[
C = 1 \text{F}; R = -1 \Omega \text{ (negative resistance)}
\]

\[
\dot{x}(t) = u(t)
\]

\[
y(t) = x(t) - u(t)
\]

for \( x(0) < 0 \), no solutions; 2 solutions for \( x(0) > 0 \)

for multiple solutions, \( x(t) > y(t) \)

\[
e.g. \text{for } x(0) = 1: \quad x(t) > y(t) = 1 \quad \text{diode}
\]

- \( u = 0 \Rightarrow x(t) = y(t) = 1 \text{; blocking diode} \)
- \( y = 0 \Rightarrow u(t) = x(t) = e^t, y(t) = 0 \text{; conducting diode} \)

**Circuit with ideal diode:**

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du \\
o &\leq y \perp u \geq 0
\end{align*}
\]

\( V(x(t)) \leq V(x(0)) + \int_0^t |u(t)| dt \)

\( \text{for continuous solutions.} \)

**Multinode:**

\[
\begin{array}{c}
\dot{x}_1 = x_2 - y_1 + u_2 \\
\dot{x}_2 = -x_1 - x_2 - y_2 \\
y_1 = -x_1 \\
y_2 = x_1 + x_2 + u_2 \\
0 \leq u \perp y \geq 0
\end{array}
\]

**4-nodes:**

\( x_1, x_2, u_1, u_2 \)

\( y_1, y_2 \) conducting current, \( y_1, y_2 \) (-ve) voltage.
In general, jumps possible in elastic circuits.

\[ \frac{d}{dt} v_c = \frac{1}{C} i_c \]

\[ i_c = \frac{1}{L} v_c \]

\[ v_c = -e \left( x \frac{dx}{dt} + h(v_c) \right) \]

\[ e = \sqrt{\frac{C}{L}} \]

Show jumps in \( v_c \) for large \( e \).

**Mechanical Systems with Unilateral Constraints**

**Hybrid**

\[ \begin{align*}
    &x_1 = x_3 \\
    &x_2 = x_4 \\
    &\dot{x}_3 = -2x_1 + x_2 \\
    &\dot{x}_4 = x_1 - x_2
\end{align*} \]

**Constrained:**

\[ \begin{align*}
    &x_1(t_0) = 0 \\
    &\dot{x}_2 = x_4 \\
    &x_3(t_0) = 0 \\
    &\dot{x}_4 = -x_2
\end{align*} \]

Event when

\[ (x_1(t_0) = 0) \land (x_3(t_0) < 0) \lor (x_2(t_0) = 0) \lor (x_4(t_0) = 0) \lor (x_4(t_0) = 0) \land (x_4(t_0) = 0)) = \text{TRUE}. \]

**Eff**

\[ \begin{align*}
    &\dot{x}_1 = -2x_1 + x_2 + \lambda \quad \text{constraint force} \\
    &\dot{x}_2 = x_1 - x_2 \\
    &0 \leq x_1 \perp \lambda > 0
\end{align*} \]

\[ x_1 = 0, \quad x_1 \leq 0, \quad \dot{x}_1 = -e \dot{x}_1 \]
General Model

In a coordinate system with $q(t)$: configuration vector, $p(t)$: vector of impulses,

$H(q, p)$: Hamiltonian $f$, generally $KE + PE$

$\frac{1}{2} p^T M(q) p + V(q)$

Unilateral constraint: $C(q(t)) \geq 0$

Equations of Motion

$\dot{q} = \frac{\partial H}{\partial p}(q, p)$

$\dot{p} = -\frac{\partial H}{\partial q}(q, p) + \frac{\partial C^T}{\partial q}(q) u$

$y = C(q)$

$0 \leq y^T u \leq 0$

Optimal Control with State Constraints

$x(t) = f(t, x(t), u(t)), \ x(0) = x_0$

$g(t, x(t), u(t)) \geq 0$

max. $\int_0^T F(t, x(t), u(t)) dt + g^T F_T(x(t))$

Define Hamiltonian $H(t, x, u, \lambda) = F(t, x, u) + \lambda^T g(t, x, u)$

Lagrangian $L(t, x, u, \lambda) = H(t, x, u, \lambda) + \eta^T g(t, x, u)$

The necessary optimal condition is:

$\dot{x}(t) = f(t, x(t))$

$\dot{\lambda}(t) = -\frac{\partial L}{\partial x}(t, x(t), u(t), \lambda(t), \eta(t))$

$u(t) = \arg \max u \ \ L(t, x(t), u(t), \lambda(t), \eta(t))$

$0 \leq g(t, x(t), u(t))^T \lambda(t) \geq 0$

BC: $x(0) = x_0, \ \lambda(T) = \frac{\partial F_T}{\partial x}(x(T))$

For LQ problem with linear constraint, we get linear Hamiltonian $CP$. 

Variable Linear System:

\[ \dot{z} = f(x, u), \quad \bar{y} = h(x, u) \]

Given feedback:

\[ \bar{u} = 1, \quad \bar{y} \geq 0 \]

\[ -1 \leq \bar{u} \leq 1, \quad \bar{y} = 0 \]

\[ \bar{u} = -1, \quad \bar{y} < 0 \]

New variables:

\[ u_1 = \frac{1}{2}(1 - \bar{u}) \quad \text{and} \quad u_2 = \frac{1}{2}(1 + \bar{u}) \]

\[ \bar{y} = y_1 - y_2 \]

Instead of \( y \), we use:

\[ y_1 = 0, \quad u_1 \geq 0 \]

\[ y_1 > 0, \quad u_1 = 0 \]

\[ y_1 > 0, \quad u_1 = 0 \quad \text{and} \quad u_1 u_2 = 1 \]

Since \( u_1 + u_2 = 1 \), there are only 3 modes here.

We can also write \( \bar{y} \) as:

\[ \dot{z} = f(x, u_2, u_1) \]

\[ y_1 - y_2 = f(x, u_2, u_1) \]

\[ u_1 + u_2 = 1 \]

Variational In = \( y \), Projected Dyn System + CP.

For a closed convex set \( K \subset R^k \) and vector \( f \in K \):

For \( k \rightarrow R^k \), determine \( x^* \in K \)

\[ \langle -F(x^*), x-x^* \rangle > 0, \quad \forall x \in K \]

\( \langle \cdot, \cdot \rangle \) means inner product in \( R^k \).

\( \text{VI}(F, K) \)

(Non-negative orthant)

CP:

\[ x^* \] solves \( \text{VI}(F, K) \) iff \( x^* \) solves the CP

Find \( x^* > 0 \)

\[ 0 \leq x^* - F(x^*) > 0 \]

PDS:

\[ \text{TK}(x, v) = \lim_{\delta \to 0} \left( F_k(x + \delta v) - x \right) \]

where \( F_k(x) = \arg \min \| x - z \| \) \( z \in K \)

when \( x \in \text{inte}(K) \), \( F_k(x + \delta v) = x + \delta v \)

\[ \text{TK}(x, v) = v \]

when \( x \in \partial K \), then \( \text{TK}(x, v) = 0 \) (see figure)
\[ x = \Pi_k (x_1, 0(x)) \text{, } x(0) = 0 \]

\[ \begin{align*}
&\dot{x}_1 = x_2 \\
&\dot{x}_2 = -4x_1
\end{align*} \]

Thus, the epimorphism of a PDS \((F_1, k)\) for \(k\) a convex polyhedron, coincide with the solutions of \(VI (F_1, k)\).

**Diffusion:**
- \(S\): price
- \(k\): exercise price (at maturity)
- \(C\): option price

E.g. a stock has current value \(S(0) = 100\), you would like to buy an option price \((C)\) if you are guaranteed at least some exercise price at maturity (European put option) or (\(k\)) some exercise price at maturity (European put option) on till maturity (American put option).

Stochastic differential equation: SDE
\[ dS(t) = \mu S(t) dt + \sigma S(t) dW(t) \]

**Weiner Process**

\[ \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \rho S \frac{\partial C}{\partial S} - r C = 0 \]

\[ C(S, 0) \text{ for European} \]

\[ C(S, T) \text{ for American} \]

\[ \leq 0 \text{ (i.e. when given fixed interest rate)} \]

\[ \text{fixed interest rate} \]

\[ \text{option not exercised} \]

\[ \text{W1 condition on} \]

\[ C(S, t) \text{, } t \\ C(0, t) = \max(k - S(t)) \text{ for put} \]

\[ \text{max}(k - S(t)) \text{ for put} \]

\[ \text{max}(k - S(t)) \text{ for call} \]
Substitute \( S = KE^2 \) ( i.e. replace \( S \) by \( x \) ); \( \nu = \frac{C}{k} \)
and introduce \( g(x) = \max (1 - x, 0) \); \( z = s - t \).

For American,
\[
\left( \frac{\partial^2 \nu}{\partial z^2} + \frac{1}{2} \sigma^2 \frac{\partial^2 \nu}{\partial x^2} + \left( v - \frac{\sigma^2}{2} \right) \frac{\partial \nu}{\partial x} + r \nu \right) \nu = 0
\]

\( \nu \geq 0, \quad m \geq 0 \)
as dimensional
complementary
system.

Break \( x \) into \( x_1, \ldots, x_N \); \( y_i \) back
in order on \( x \)

\[
x = \begin{bmatrix} A & 0 \\ 0 & \infty \end{bmatrix} x + \begin{bmatrix} I \\ 0 \end{bmatrix} u, \quad x = \begin{bmatrix} u \\ 1 \end{bmatrix}
\]

\( y = [I - g] x \)

\( 0 \leq y \perp u \geq 0 \)

We can also use linear inequalities

Max - plus system. (used in Petri nets)

\( z = \max (x, y) \)

Also as \( z = x + a \geq y + b \),

\( 0 \leq a \perp b \geq 0 \).
Existence and Uniqueness of Solutions

In general, necessary & sufficient conditions for existence of solutions of hybrid systems: different. For many classes of problems, we can provide some necessary conditions.

Uniqueness: (Right uniqueness), i.e. given \( x(t) \),

only \( x \in [t_0, t] \).

In CP, e.g. two cart example with \( e = 0 \).

Local Existence: If solution exists for \([t_0, t_0 + \epsilon] \).

Global: \( \phi \) on \([t_0, \infty) \).

Local uniqueness for all initial conditions + global existence \( \Rightarrow \) global uniqueness.

Local existence \( \Rightarrow \) global existence.

E.g. \( \dot{x} = x^2 \), \( x(0) = x_0 \) gives \( x(t) = x_0 (1 - x_0 t)^{-1} \)
defined for \([0, x_0^{-1}] \).

Hybrid systems: Accumulation of mode switches (Zeno)

E.g. bouncing ball.

Also e.g. Fuller phenomenon: \( \dot{x}_1 = \dot{x}_2 \)
\( \ddot{x}_2 = 4 \), \( x(0) = 0 \)

\( |x(t)| \leq 1 \)

\( \int_{t_0}^{t} |x(t)| dt \)

Long-run \omega \) as switches.
Mode selection problem

"smooth continuation"

\[
\begin{align*}
\dot{x} &= f(x, u) \\
y &= h(x, u) \\
0 \leq y + u \geq 0
\end{align*}
\]

\[
y(0) = h(x(0), u(0)) \\
y = \frac{\partial y}{\partial x} \begin{pmatrix} \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u} \\
\frac{\partial y}{\partial x}
\end{pmatrix}
\]

\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{y_1(x_1, x_2)}{x_1^2 + x_2^2} dx_1 dx_2 = \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{y_2(x_1, x_2)}{x_1^2 + x_2^2} dx_1 dx_2
\]

Input affine system

\[
\begin{align*}
\dot{x}(t) &= g(x(t), u(t)) \\
y(t) &= h(x(t))
\end{align*}
\]

Relative degree of \( y_t \) is the number of differentiations to get \( u \).

Constant uniform rel. deg. at \( x_0 \), if rel. deg. of all \( y_t \) are the same and are constant in the neighborhood of \( x_0 \).

Thus \( y_t \) is \( \xi \) solution of \( x_{t+1} = f(x_t, u_t) \) where \( u_t \) is feasible set.

Proof:

\[
\begin{align*}
\dot{x}_1 &= -x_2 u \\
\dot{x}_2 &= 1 - x_1 u \\
y &= -x_2 \\
0 \leq y + u \geq 0
\end{align*}
\]

For \( y = 0 \), \( \dot{y} = 0 \) \( x_1^2 + x_2^2 = x_1 + x_2 \)
A. Representing electric networks, we derive, the system is "passive" and well posed.
(proof 95-96)

Passive: \( \exists a \) s.t. \( V(x) > 0 \) s.t.
\[
L_f V(x) < 0 \\
L_g_i V(x) = h_i(x), i = 1, ..., k
\]
Assume (non-degeneracy)
\[
\text{rank } \left[ L_g_i L_g_i V(x) \right] = k \quad \forall x \text{ w/ } h(x) > 0
\]
Then \( L_g_i h_i = L_f L_g_i V \) (\( \therefore \) uniform sol. def.: 1)
Decoupling matrix \( D(x) \)
If vectors \( g_i \) are commuting \( \therefore [g_i, g_j] = 0 \)
Then \( D(x) \) is symmetric.
\[
\Rightarrow D(x) = D(x)^T > 0, \quad \text{DCP solved.}
\]

For DCP
\[
\dot{x} = Ax + Bu, \quad y = Cx + Du \\
0 \leq y \leq 30
\]
For \( k = 1, \) transfer function
\[
g(s) = g_0 + g_1 s + g_2 s^2 + \ldots
\]
\[
= C(s^2 - A)^{-1} B + D\quad \text{rational}
\]
\( g_i \): numerator parameter
\( C \neq 0, \) (otherwise triv y = 0 + see \( \dot{x} = Ax, \) not bimodal)

The leading numerator parameter will be the first \( g_0 \) (\( C = 0, 1, 2 \ldots \))
matrix version for the general case.
Look at
\[ y(s) = y^0 s^{-1} + y^1 s^{-2} + y^2 s^{-3} + \cdots \]
\[ (y^0, y^1, \ldots)^T = y(s) \geq 0 \text{ for } s \text{ large} \]

If \( y(s) \) and \( u(s) \) are related as
\[ y(s) = C (sI - A)^{-1} x_0 + (D + C (sI - A)^{-1} B) u(s) \]
then
\[ 0 \leq y(t) \leq y \]

\[ RCP \]
\[ y(s) = T(s)x_0 + C(s)u(s) \]
\[ \text{for all } s \geq 0 \]
\[ 0 \leq y(s) \leq y \]

\[ RCP \text{ was a mistake if } LDCP \text{ was a mistake} \]

Mode Selection:
\[ x_1 = -2x_1 + x_2 + u(t) \]
\[ x_2 = x_1(t) - x_2(t) \]
\[ y(t) = x_1(t); \quad 0 \leq y(t) \leq y \]

\[ (s^2 + 1)x_1 = x_2 + u + s^2 \]
\[ (s^2 + 1)x_2 = x_1 + x_4 + s^2 \]

\[ (s^2 + 1)(s^2 + 1)x_1 = x_1 + x_4 + s^2 x_2 + (s^2 + 1)(u + x_3 + s^2 x_1) \]

\[ (s^4 + 3s^2 + 1)x_1 = \begin{bmatrix} s(s^2 + 1), s, s^2 + 1, 1 \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{14} \\ x_2 \\ x_4 \end{bmatrix} + (s^2 + 1)u \]

If \( y = x + u; \ y > 0, u \geq 0, y u = 0 \)
\[ \begin{align*}
&x > 0, u = 0; y = x \\
&x < 0, u = -x \\
&x = 0, y = u = 0
\end{align*} \]

at collision \( x_{10} = 0 \) \( \Rightarrow \) highest order term \( x_{30} \)
\[ \begin{align*}
&\text{if } x_{30} > 0 \text{ then } u = 0 \\
&\text{if } x_{30} < 0 \text{ then constraint comes in} \\
&\text{if } x_{30} = 0 \text{ then look at } x_{20} \text{ term}
\end{align*} \]

etc.
3.2. Example

respect to the stop. The force exerted by the stop can only act in the positive direction implying that \( u(t) \) should be nonnegative. If the left cart is not at the stop at time \( t \) \((y(t) > 0)\), the reaction force vanishes at time \( t \), i.e. \( u(t) = 0 \). Similarly, if \( u(t) > 0 \), the cart must necessarily be at the stop, i.e. \( y(t) = 0 \). This is expressed by the conditions

\[
0 \leq y(t) L u(t) \geq 0.
\]  

(3.2)

The system can be represented by two modes, depending on whether the stop is active or not. We distinguish between the unconstrained mode \((u(t) = 0)\) and the constrained mode \((y(t) = 0)\). The dynamics of these modes are given by the following Differential and Algebraic Equations (DAEs)

\[
\begin{align*}
\text{unconstrained} & \quad \text{constrained} \\
\dot{x}_1(t) &= x_3(t) & \dot{x}_1(t) &= x_3(t) \\
\dot{x}_2(t) &= x_4(t) & \dot{x}_2(t) &= x_4(t) \\
\dot{x}_3(t) &= -2x_1(t) + x_2(t) & \dot{x}_3(t) &= -2x_1(t) + x_2(t) + u(t) \\
\dot{x}_4(t) &= x_1(t) + x_2(t) & \dot{x}_4(t) &= x_1(t) + x_2(t) \\
u(t) &= 0 & y(t) &= x_1(t) = 0.
\end{align*}
\]

When the system is represented by either of these modes, the triple \((u, x, y)\) is given by the corresponding dynamics as long as the inequalities in (3.2)

\[
\begin{align*}
\text{unconstrained} & \quad \text{constrained} \\
y(t) \geq 0 & \quad u(t) \geq 0
\end{align*}
\]

are satisfied. A mode change is triggered by violation of one of these inequalities. The mode transitions that are possible for the two-carts systems are described below.

- **Unconstrained → Constrained**: The inequality \( y(t) \geq 0 \) tends to get violated at a time instant \( t = \tau \). The left cart hits the stop and stays there. The velocity of the left cart is reduced to zero instantaneously at the time of impact: the kinetic energy of the left cart is totally absorbed by the stop due to a purely inelastic collision. A state for which this happens is, for instance, \( x(\tau) = (0, -1, -1, 0)^T \).

- **Constrained → Unconstrained**: The inequality \( u(t) \geq 0 \) tends to be violated at \( t = \tau \). The right cart is located at or moving to the right of its equilibrium position, so the spring between the carts is stretched and pulls the left cart away from the stop. This happens for example if \( x(\tau) = (0, 0, 0, 1)^T \).

- **Unconstrained → Unconstrained with re-initialization according to constrained mode**: The inequality \( y(t) \geq 0 \) tends to get violated at \( t = \tau \). As an example, consider \( x(\tau) = (0, 1, -1, 0)^T \). At the time of impact, the velocity
of the left cart is reduced to zero just as in the first case. Hence, a state jump (re-initialization) to $(0, 1, 0, 0)^T$ occurs. The right cart is at the right of its equilibrium position and pulls the left cart away from the stop. Stated differently, from $(0, 1, 0, 0)^T$ smooth continuation in the unconstrained mode is possible.

This last transition is a special one in the sense that first the constrained mode is active causing the corresponding state jump. After the jump no smooth continuation is possible in the constrained mode resulting in a second mode change back to the unconstrained mode.

From state $x(\tau) = (0, -1, -1, 0)^T$, we can enter the constrained mode by starting with an instantaneous jump to $x(\tau^+) = (0, -1, 0, 0)^T$. This jump can be modelled as the result of a (Dirac) pulse $\delta$ exerted by the stop. In fact, $u = \delta$ results in the state jump $x(\tau^+) - x(\tau) = (0, 0, 1, 0)^T$. This motivates the use of distributional theory as a suitable mathematical framework for describing physical phenomena like collisions with discontinuities in the state vector.

To summarize, the motion of the carts is governed by two systems of Differential and Algebraic Equations (DAEs), called the constrained and the unconstrained mode. A change of mode is triggered by violation of certain inequalities corresponding to the current mode. The time instants at which this occurs, are called "event times." At an event time, the system will switch to a new mode. A mode transition often calls for a state jump or re-initialization. In the example, velocity jumps occur, when the left cart arrives at the stop with negative velocity. In this chapter, the above dynamics will be formalized for the complete class of linear complementarity systems and special attention will be paid to the mode selection problem and well-posedness issues. However, first we recall some facts concerning systems of linear differential and algebraic equations, such as appear in the constrained and unconstrained mode descriptions.

### 3.3 Mathematical Preliminaries

We consider a linear differential/algebraic system of the form

$$\begin{align*}
\dot{x}(t) &= Kx(t) + Lu(t) \quad (3.3a) \\
0 &= Mx(t) + Nu(t). \quad (3.3b)
\end{align*}$$

The time arguments will often be suppressed for brevity. Throughout this section, $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. The system parameters $K$, $L$, $M$ and $N$ are constant matrices of dimensions $n \times n$, $n \times m$, $r \times n$ and $r \times m$, respectively.

**Definition 3.3.1** A state $x_0$ is said to be consistent for $(K, L, M, N)$, if there exist smooth functions $u$ and $x$ such that $x(0) = x_0$ and (3.3) is satisfied. The set of all consistent states for $(K, L, M, N)$ is denoted by $V(K, L, M, N)$ and is called the consistent subspace.
written as \((u^1, u^2, \ldots, u^j) \geq 0\) if \((u^1, u^2, \ldots, u^j) = (0, 0, \ldots, 0)\) or \(u^j > 0\) where
\(j := \min\{p \in \mathbb{N} \mid u^p \neq 0\}\). A sequence of scalars is called lexicographically positive, denoted by \((u^1, u^2, \ldots, u^j) > 0\), if \((u^1, u^2, \ldots, u^j) \geq 0\) and \((u^1, u^2, \ldots, u^j) \neq (0, 0, \ldots, 0).\) For a sequence of vectors \((u^1, u^2, \ldots, u^j)\) with \(u^j \in \mathbb{R}^k\), we write \((u^1, u^2, \ldots, u^j) \geq 0\) when \((u^1, u^2, \ldots, u^j) \geq 0\) for all \(i \in \mathbb{N}\). Likewise, we write \((u^1, u^2, \ldots, u^j) > 0\) when \((u^1, u^2, \ldots, u^j) > 0\) for all \(i \in \mathbb{N}\).

For sets \(A\) and \(B\), \(A \setminus B := \{x \in A \mid x \notin B\}\) and \(P(A)\) denotes the power set of \(A\), i.e. the collection of all subsets of \(A\). For two subspaces \(V, T\) of \(\mathbb{R}^n\), the notation \(V \oplus T = \mathbb{R}^n\) means that \(V\) and \(T\) form a direct sum decomposition of \(\mathbb{R}^n\), i.e. \(V + T := \{v + t \mid v \in V, \ t \in T\} = \mathbb{R}^n\) and \(V \cap T = \{0\}\).

3.2 Example

Before specifying the class of linear complementarity systems (LCS), we illustrate some of the aspects that play a role in the evolution of such systems by an example of two carts connected by a spring (used also in [177]). The left cart is attached to a wall by a spring. The motion of the left cart is constrained by a completely inelastic stop. The system is depicted in figure 3.1.

![Figure 3.1: Two-carts system.](image)

For simplicity, the masses of the carts and the spring constants are set equal to 1. The stop is placed at the equilibrium position of the left cart. By \(x_1, x_2\) we denote the deviations of the left and right cart, respectively, from their equilibrium positions and \(x_3, x_4\) are the velocities of the left and right cart, respectively. By \(u\), we denote the reaction force exerted by the stop. Furthermore, the variable \(y\) is set equal to \(x_1\).

Simple mechanical laws lead to the dynamical relations

\[
\begin{align*}
\dot{x}_1(t) &= x_3(t) \\
\dot{x}_2(t) &= x_4(t) \\
\dot{x}_3(t) &= -2x_1(t) + x_2(t) + u(t) \\
\dot{x}_4(t) &= x_3(t) - x_2(t) \\
y(t) &:= x_1(t).
\end{align*}
\] (3.1)

To model the stop in this setting, the following reasoning applies. The variable \(y(t) = x_1(t)\) should be nonnegative, because it is the position of the left cart with
Forward Zero Problems

\[ \dot{x}_1 = -5f_u(x_1) + 2f_u(x_2) \]
\[ \dot{x}_2 = -2f_u(x_1) - 5f_u(x_2) \]

\[ \frac{d}{dt} \left( |x_1(t)| + |x_2(t)| \right) = -2 \quad \text{when} \quad x(t) \neq 0 \]

\[ \dot{x}_1 = 5f_u(x_1) - 2f_u(x_2) \]
\[ \dot{x}_2 = 2f_u(x_1) + 5f_u(x_2) \]

Problem 1: multiple solutions at reversed time

Problem 2: How do you continue the solution forward? How do you match the first event?