Berkeley Lectures:

Mathematical Principles for Engineering and Applied Sciences done Visually

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Contents

CHAPTER 1

EXISTENCE OF OPTIMAL SOLUTIONS

In this lecture, we start with studying basic optimization problems, and look at when the solutions exist for the problems. Then we expand the spaces in the abstract sense to generalize the results, and in doing so, we realize the importance of *topology*. Optimization is mostly carried out over real numbers because the numbers can be compared, i.e. there is an order in the set of real numbers. Hence the cost function maps from its domain to real numbers, and we want to pick that member of the domain whose cost is minimum or maximum.

1.1 Optimization over an Interval

Consider minimizing a real valued function as shown in Figure 1.1. The left most case considers the function $f : (0,1) \to \mathscr{R}$ given by $f(x) = x$. There is no $x_0 \in (0,1)$ such that $f(x_0) \leq f(x) \forall x \in (0,1)$, since for any $x_0 \in (0,1)$ we take we can find $x_* \in (0,1)$ so that $f(x_*) < f(x_0)$. This is easily accomplished by taking for instance $x_* = x_0/2$. The function $f(x) = x$ is continuous. We note a special property of the set $(0,1)$, which is that each point of this set is an internal point. Specifically for every point $x \in (0,1)$, we can find an open interval I such that $x \in I$ and $I \subset (0,1)$. What we mean by an open interval is the existence of two real numbers a and b, such that $0 < a < b < 1$ and $x \in (a,b)$. Hence, there is no point $x \in \mathscr{D}$ ($\mathscr{D} = (0,1)$ here is the domain), which achieves a value that is lower than all other values achieved by the function in its domain.

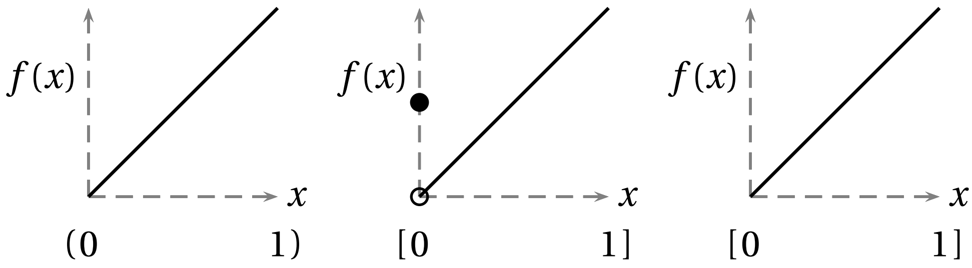

Figure 1.1: Minimizing Function Over Domains

Let's consider the case of the function shown in the middle plot in Figure 1.1. This function is given by

$$f(x) = \begin{cases} x & x \in (0,1] \\ 0.5 & x = 0 \end{cases} \tag{1.1}$$

Notice that for this function now, we have fixed the domain. So that the function is $f : [0,1] \to \mathscr{R}$ and is given by expression 1.1. In this case also there is no $x_0 \in [0,1]$ such that $f(x_0) \leq f(x) \forall x \in [0,1]$. The point $x = 0$ does not provide a minimum, and starting from any other point, we can find a point to the left that has a lower value. Hence, in this case as well, there is no point $x \in \mathscr{D}$ ($\mathscr{D} = (0,1)$ here is the domain), which achieves a value that is lower than all other values achieved by the function in its domain. In this case the domain is not open (in fact, it is closed and bounded), but the function is not continuous.

Now let's consider the case of the function shown in the rightmost plot in Figure 1.1. Here, the function $f : \mathscr{D} \to \mathscr{R}$, where $\mathscr{D} = [0,1]$ is closed and bounded and the function given by $f(x) = x$ is continuous. We see that for this case, the minimizing point $x = 0$ exists in the domain of the function.

This result is a special case of the Weierstrass theorem ([Rudin(1976)] and [Luenberger(1968)]). The statement of the Weierstrass theorem is as follows.

Theorem 1.1.1. ***Weierstrass theorem****: A continuous real valued function on a compact set achieves its maximum and minimum values.*

This chapter will be devoted to understanding the terms and the proof of this theorem. According to the statement it seems the two important aspects are *continuous function* and *compact domain.* We will understand these first in the context of the examples we started with, where the domain is the subset of the set of real numbers $\mathscr{R}$. Following that we will take the most available general view available.

In our examples, the set $[0,1]$ is a compact subset of $\mathscr{R}$ and the function $f(x) = x$ is continuous, and hence the function achieves it minimum at $x = 0$ and maximum at $x = 1$.

This theorem statement, in fact, is too strong. The minimum and maximum requires less than continuity. These theorems are listed below and we will start to understand the terms and details of these two theorems as well as we proceed.

Theorem 1.1.2. *A lower semicontinuous real valued function on a compact set achieves its minimum value.*

Theorem 1.1.3. *An upper semicontinuous real valued function on a compact set achieves its maximum value.*

Figure 1.2 shows example of a lower semicontinuous function achieving its minimum on the left and an upper semicontinuous function achieving its maximum on the right.

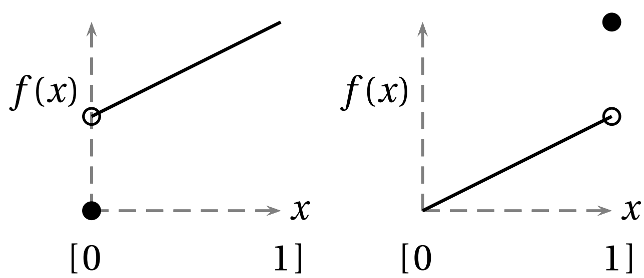

Figure 1.2: Semicontinuous Functions Achieveing Optimum Values

1.2 The Set of Real Numbers $\mathscr{R}$

We will be studying the optimization examples that we introduced in section 1.1. These problems had the domain a subset of real numbers. The results were very intricately related to the properties of the set of real numbers. Hence, we will study this set in this section. We will build the set $\mathscr{R}$ from the set of natural numbers $\mathcal{N}$. We present the Peano's axioms next to *create* the set $\mathcal{N}$ (see [Landau(2001)]).

The Set of Natural Numbers $\mathcal{N}$

Peano's axioms present the formalization of the set of natural numbers. It essentially says that

1. The number 1 is a natural number.
2. Each natural number has a unique successor. We will use $s(k)$ to denote the successor of k. After the operation addition with the symbol + is defined, the successor can be defined as $k+1$.
3. 1 is not a successor of any natural number.
4. Every natural number is a successor of at most one natural number.
5. If a set contains 1, and if containing a natural number n implies containing $s(n)$, then the set contains all natural numbers. This is called the law of mathematical induction.

More formally, Peano's axioms that develop the set $\mathcal{N}$ are:

Axiom 1: $1 \in \mathcal{N}$.

Axiom 2: $m = n \Rightarrow s(m) = s(n)$

Axiom 3: $\forall n \in \mathcal{N}, s(n) \neq 1$

Axiom 4: $s(m) = s(n) \Rightarrow m = n$

Axiom 5: $(1 \in \Omega \text{ and } ((n \in \Omega) \Rightarrow (s(n) \Rightarrow \Omega))) \Rightarrow \mathcal{N} \subset \Omega$

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[Landau(2001)] Edmund Landau. *Foundations of Analysis*. Chelsea Publishing Company, third revised edition, 2001.

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