

# Travel Time Dynamics for Transportation Systems: Theory and Applications

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## Abstract

This paper presents the various transportation application problems where travel time is an important variable and a factor. The paper demonstrates the limitation of the flow based travel time functions. The paper presents a density based travel time function. Following this, the paper develops a fundamental theory of travel time dynamics that is built from a given fundamental traffic relationship and vehicle characteristics. The travel time dynamics produce an asymmetric one-sided coupled system of hyperbolic partial differential equations, where the first equation represents the macroscopic traffic dynamics. The existence of the solution for the mathematical model is then presented.

## 1 Introduction

Transportation systems are characterized by the two main aspects: safety and throughput. From the throughput aspect, travel time in transportation networks is the most important performance index of the system operation. The aim of most transportation systems is to provide a safe, comfortable minimum transit time between any two points (nodes) of the network. An exception might be the case when a user is driving, for instance, for purely as a recreational activity, and would like to travel slow while watching the surrounding scenery.

Travel time is used in many design and operations problems in transportation. For instance, it is used to perform traffic assignment ([3]), whether that is static assignment ([33]) or dynamic traffic assignment ([22]). Feedback control based dynamic traffic assignment is also based on travel time ([16],[30],[19],[20],[17]). Travel time is an essential element of transportation planning models ([1],[9]). Similarly, it is used to develop performance measures for before and after events studies. It is also used to monitor the performance of the system in real time, as well as for aggregate performance over a time period for decision making and policy considerations.

Various methods of travel time data collection are presented in a Federal Highway (FHWA) report ([36]). Travel time prediction has been studied by many researchers and is extremely useful in many applications([2], [24], [8]).

The original travel time paper developing the travel time model based on the traffic fundamental diagram is [21]. That model was polished and a partial differential equation for travel time dynamics was presented in [22]. This paper presents the theoretical existence proof for this mathematical travel time dynamics model, and then presents some applications where the model can be implemented.

**Outline** The remainder of this paper is organized as follows. We review the travel time function used in practice in Section 2, where in Subsection 2.1 we show its application in static traffic assignment, in Subsection 2.2 we show some other applications, and in Subsection 2.3 we study the limitations of the static travel time functions. We introduce the travel time dynamics integrated with macroscopic traffic model in Section 3, where in Subsection 3.3, we present its derivation, and in Subsection 3.4, we present the proof of the existence of its solutions and various properties.

## 2 Static Travel Time Function and its Uses

The static travel time function  $T(f)$  is a function of traffic flow  $f$  (also called traffic volume) on the link and the link capacity  $C$ . Capacity is defined as the traffic flow at traffic density whose value is half of the traffic jam density. Traffic jam density is the traffic density at which the vehicle speed is zero. It is also called maximum density. The static travel time function  $T(f)$  is given by Equation 1

$$T(f) = t_f \phi \left( \frac{f}{C} \right) \quad (1)$$

where  $t_f$  is the freeflow travel time of a vehicle on the link, i.e. the time taken by a vehicle to traverse the link when traffic density on the link is zero.

The formula Bureau of Public Roads (BPR) gives uses a specific function  $\phi(\cdot)$ . Their model is given by the Equation 7

$$T(f) = t_f \left( 1 + \beta \left( \frac{f}{C} \right)^\alpha \right) \quad (2)$$

where there are two parameters  $\beta$  usually taken as 1, and  $\alpha$  whose value usually ranges from 2 to 12 in practice. The plot of the BPR function is shown in Figure 1

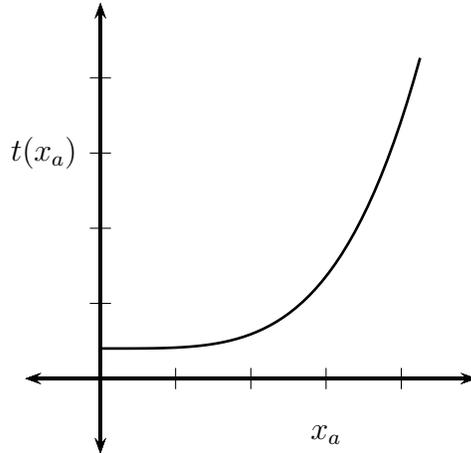


Figure 1: BPR Link Performance Function

General properties that a travel time function on a link should satisfy in this formulation are presented in [34]. Some of these are:

- **Continuous Upto Second Differentiability:**  $T(\cdot) \in C^2$
- **Positivity:**  $\forall f \geq 0, T(f) \geq 0$

- **Monotonicity:**  $f_1 \geq f_2 \Rightarrow T(f_1) \geq T(f_2)$
- **Strict Monotonicity of Slope:**  $f'' > 0$
- **Boundedness of Slope:**  $\exists M > 0, f' \leq M$
- **Uniqueness of Link Volume:**  $f'(0) > 0$

Many travel time functions have been proposed that satisfy some or all of these requirements ([34],[4], [15])

## 2.1 Traffic Assignment

To build the mathematical framework for this section, we will start with terminology and framework used in [33]. We illustrate a sample network that is also taken from [33] and is shown in Figure 2. The digraph shows four nodes and four arcs. Nodes 1 and 2 are origin nodes and node 4 is the destination node. Hence there are two O-D pairs: 1 – 4 and 2 – 4.

Table 1: Network Notation

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$\mathfrak{N}$	Set of Nodes
$\mathfrak{A}$	Set of Arcs
$\mathfrak{R}$	Set of Origin Nodes
$\mathfrak{S}$	Set of Destination Nodes
$\mathfrak{K}$	Set of Paths connecting O-D pair $r - s$ , $r \in \mathfrak{R}$ , $s \in \mathfrak{S}$
$x_a$	Flow on arc $a \in \mathfrak{A}$
$t_a$	Travel time on arc $a \in \mathfrak{A}$
$f_k^{rs}$	Flow on path $k \in \mathfrak{K}$ between O-D pair $r - s$
$c_k^{rs}$	Travel time on path $k \in \mathfrak{K}$ between O-D pair $r - s$
$q_{rs}$	O-D Trip rate between O-D pair $r - s$
$\delta_{a,k}^{rs}$	$\delta_{a,k}^{rs} = 1$ , if $a$ is in path $k$ between $r$ and $s$ , otherwise 0

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There are two main classical traffic assignment optimization problems considered. Those two are: user-equilibrium, and system optimum.

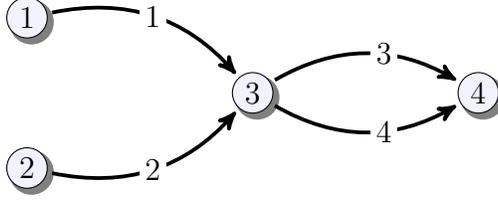


Figure 2: Sample Network

### 2.1.1 User-equilibrium

User-equilibrium problem is based on Wardrop's principle [38] which is stated as:

*The journey times on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.*

This equilibrium condition can be obtained as a solution of a mathematical programming problem presented below [33].

**Mathematical Programming Formulation** The user equilibrium problem is stated as the mathematical programming problem (see [33], [6]) shown in Equation 3.

$$\min z(x) = \sum_a \int_0^{x_a} t_a(\omega) d\omega \quad (3)$$

with the equality constraints:

$$\sum_k f_k^{rs} = q_{rs} \quad \forall r, s \quad (4)$$

$$x_a = \sum_r \sum_s \sum_k f_k^{rs} \delta_{a,k}^{rs} \quad (5)$$

and the inequality constraint

$$f_k^{rs} \geq 0 \quad \forall r, s \quad (6)$$

The formulation given in Equation 3 is the Beckmann transformation [3]. The link performance function  $t_a(x_a)$  is a function of traffic flow on the link

and the link capacity  $c_a$ . According to the Bureau of Public Roads (BPR) it is given by Equation 7

$$t_a(x_a) = v_f \left( 1 + 0.15 \left( \frac{x_a}{c_a} \right)^4 \right) \quad (7)$$

### 2.1.2 System Optimal Solution

System optimal solution is a solution that provides the total minimum time for the entire network. This condition can be obtained as a solution of a mathematical programming problem presented below [33].

**Mathematical Programming Formulation** The system optimal problem is stated as the mathematical programming problem (see [33], [6]) shown in Equation 8.

$$\min z(x) = \sum_a x_a t_a(x_a) \quad (8)$$

with the equality constraints:

$$\sum_k f_k^{rs} = q_{rs} \quad \forall r, s \quad (9)$$

$$x_a = \sum_r \sum_s \sum_k f_k^{rs} \delta_{a,k}^{rs} \quad (10)$$

and the inequality constraint

$$f_k^{rs} \geq 0 \quad \forall r, s \quad (11)$$

## 2.2 Others

There are many applications where travel time is a crucial parameter, variable, or an index which is used for design and/or evaluation of the system. For instance, we can design the timing sequencing for traffic lights at signalized intersections to minimize travel time between all possible nodes. In Figure 3, we see the nodes  $n_1$  to  $n_6$ , and two sets of traffic lights. The sequencing for traffic lights at the two signalized intersections can be designed and synchronized such that the sum of all the travel time between each pair

of these nodes is minimized. The travel time between any pair will be the sum of the link travel time and the queue delay. This paper is only concerned with the theory of link travel time.

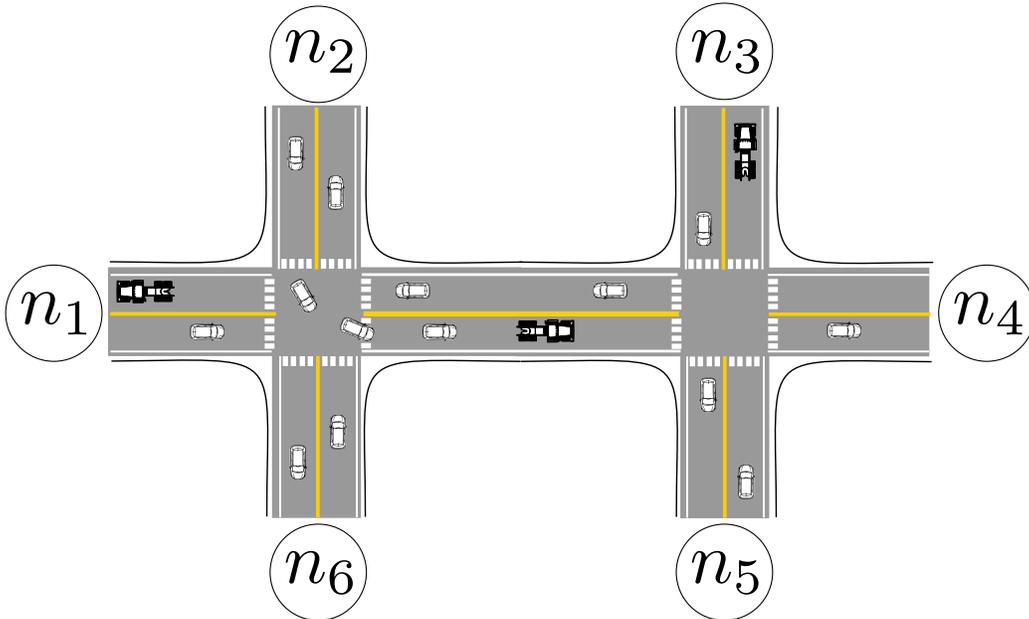


Figure 3: Travel Time between Various Nodes

### 2.3 Limitations of the Static Travel Time Functions

In order to study the limitations of the static travel time functions, we will first present the traffic fundamental diagram and relationships, which will be the foundation for developing the theory of travel time dynamics. There are many different models that related the traffic density  $\rho$   $f$  to the traffic flow (or flux or volume) which is the product of traffic density and the traffic speed  $v$ , i.e.  $f = \rho v$ . Some models use linear relationship (Greenshield's model, [12]), logarithmic relationship (Greenberg model, [11]), exponential relationship (Underwood model, [37]), piecewise linear relationship (cell transmission model, [7]), and many others, such as Northwestern University model, Drew model, Pipes-Munjal model, and multi-regime models. However, because of its mathematical simplicity we will use the Greenshield's model for developing

the theory. The theory can be modified if we choose any other relationship.

### 2.3.1 Fundamental Diagram

Greenshield's model (see [12]) uses a linear relationship between traffic density and traffic speed.

$$v(\rho) = v_f \left(1 - \frac{\rho}{\rho_m}\right) \quad (12)$$

where  $v_f$  is the free flow speed and  $\rho_m$  is the maximum density. Free flow speed is the speed of traffic when the density is zero. This is the maximum speed. The maximum density is the density at which there is a traffic jam and the speed is equal to zero.

The traffic flow is the product of traffic density and speed and becomes

$$f(\rho) = v_f \rho \left(1 - \frac{\rho}{\rho_m}\right) \quad (13)$$

The traffic flow function (of density) is concave as can be confirmed by noting the negative sign of the second derivative of flow with respect to density, i.e.  $\partial^2 f / \partial \rho^2 < 0$ . The fundamental diagram refers to the relationship that the traffic density  $\rho$ , traffic speed  $v$  and traffic flow  $f$  have with each other. These relationships are shown in Figure 4.

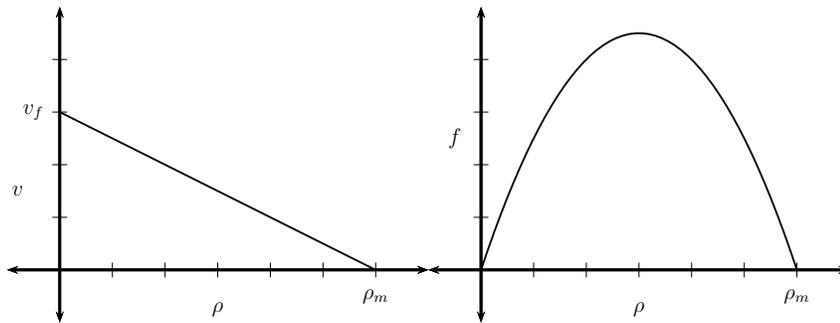


Figure 4: Fundamental Diagram using Greenshield Model

### 2.3.2 Travel Time based on the Fundamental Diagram

To study the fundamentals of travel time, let us take a single highway stretch as shown in Figure 5. The length of the stretch is  $\ell$ . We show the two ends of the stretch as  $n_1$  and  $n_2$ , and the traffic density is a function of time  $t$  and (one dimensional) space  $x$ , so that we show density as  $\rho(t, x)$ . In steady state conditions, the traffic conditions don't change over time. Moreover, let us assume that the density is same over the entire section. Let's call this fixed value of density  $\rho_0$ .

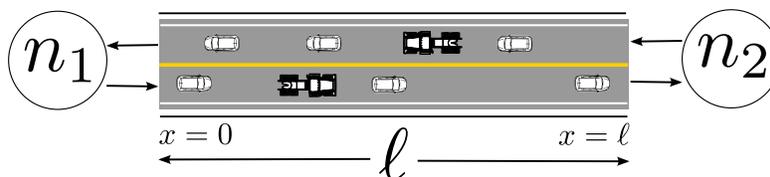


Figure 5: Travel Time over a Highway Section

The speed anywhere in this section will be constant, as it is given by the formula

$$v(\rho) = v_f \left(1 - \frac{\rho_0}{\rho_m}\right) \quad (14)$$

Hence, the travel time over the link which should be the distance of the link divided by the speed, is given by

$$T = \frac{\ell}{v_f \left(1 - \frac{\rho_0}{\rho_m}\right)} \quad (15)$$

Notice that the capacity of a link was given by the traffic flow corresponding to the maximum (or jam) density. Hence, for the linear model we have chosen, the capacity turns out to be maximum flow given by:

$$C = \frac{1}{4} v_f \rho_m \quad (16)$$

This travel time function is a function of the speed as compared to the other functions that were functions of traffic flow. The plot of this travel time function is shown in Figure 6.

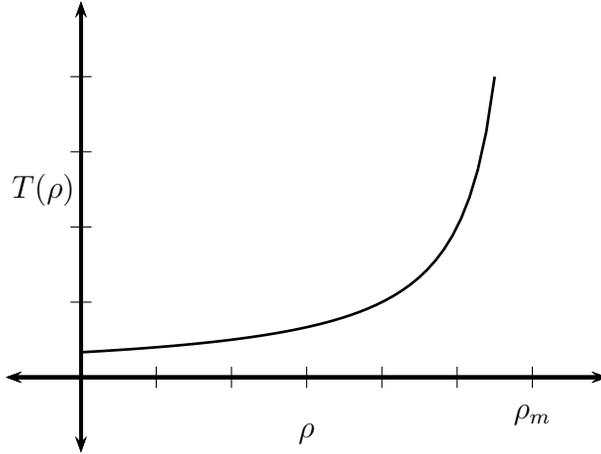


Figure 6: Travel Time Function based on Density

The highway stretch is in equilibrium (steady state). The flow into the highway from node  $n_1$  is equal to the flow out from the section to the node  $n_2$ , and is given by:

$$f = v_f \rho_0 \left(1 - \frac{\rho_0}{\rho_m}\right) \quad (17)$$

Now, let us say, just like in traffic assignment problems where the travel time functions of traffic flow are used, that we are given a steady state traffic flow in going from  $n_1$  to  $n_2$  that equals  $f_0$ . For this given traffic flow, there will be two possible densities that produce the same flow. For this given flow, let us name the two densities  $\rho_{\ell_0}$  and  $\rho_{r_0}$  as shown in Figure 7.

Hence, for the same fixed steady state traffic flow given, there are two different travel times that are consistent with that data. They are given by Equation 18.

$$T(\rho_{\ell_0}) = \frac{\ell}{v_f \left(1 - \frac{\rho_{\ell_0}}{\rho_m}\right)} \quad T(\rho_{r_0}) = \frac{\ell}{v_f \left(1 - \frac{\rho_{r_0}}{\rho_m}\right)} \quad (18)$$

Although the travel time given by the function based on traffic flows is single valued, but that function is not consistent with the theory that a

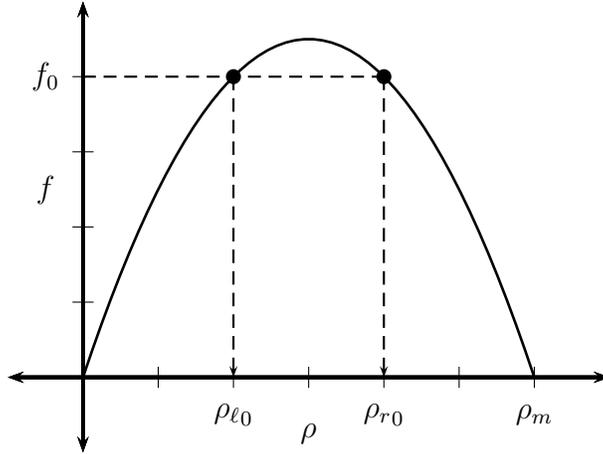


Figure 7: Two Densities for the Given Traffic Flow

vehicle with the speed  $v(\rho)$  will traverse the link in time given by the ratio of the distance to the speed. Hence, the problem of estimating travel time or even performing traffic assignment based on flow based travel time functions is an ill-posed problem. The problem must provide density information.

### 2.3.3 Summary of the Limitations

Hence, there are essentially two limitations of the travel time functions based on traffic flow. These are:

1. The travel time functions are not consistent with travel time obtained by a vehicle traveling with speed consistent with the density corresponding to the flow from the fundamental diagram.
2. The travel time functions are valid for only steady state conditions.

Using the travel time function based on the traffic flow theory by Equation 17, the first issue has been addressed. To address the second issue, we proceed to develop a model that would give travel time considering spacial and temporal dynamics.

### 3 Travel Time Dynamics

In order to develop the travel time dynamics, we need to use a corresponding consistent traffic model. We use the Lighthill-Whitham-Richards (LWR) model, named after the Lighthill, Whitham, and Richards in [28] and [31], which is a macroscopic one-dimensional traffic model. The conservation law for traffic in one dimension is given by

$$\frac{\partial}{\partial t}\rho(t, x) + \frac{\partial}{\partial x}f(\rho(t, x)) = 0 \quad (19)$$

#### 3.1 Generalized/Weak Solution for the LWR Model

The hyperbolic Partial Differential Equation (PDE) for the LWR model given by Equation 19 can be solved by using the method of characteristics ([27]). For details on the method of characteristics for the traffic problem, please refer to part three of the book [13] and chapter 5 of the book [18]. We present just the main result of solving the Riemann problem using the method of characteristics next.

##### 3.1.1 Riemann Problem Solutions

Riemann problem is solving the LWR model where the initial condition is a piecewise constant function with two values  $\rho_\ell$  and  $\rho_r$  for the upstream (left) and downstream (right) densities (see [26]). Solution of the Riemann problem leads to theoretical proofs as well as numerical techniques to solve the hyperbolic PDE problems. From the junction of the two densities either a shockwave or a rarefaction wave can emanate. A shockwave develops if  $f'(\rho_\ell) > f'(\rho_r)$  (see [25]).

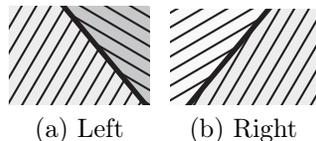


Figure 8: Shockwaves moving Upstream (left) and Downstream (right)

The speed of the shockwave is given by Equation (20). In this equation,  $x_s(t)$  is the position of the shockwave as a function of time. If the shock

speed is positive then the inflow at junction between the two traffic densities will be a function of upstream traffic density, whereas if the shock speed is negative then the inflow at junction between the two traffic densities will be a function of downstream traffic density.

$$s = \frac{dx_s(t)}{dt} = \frac{[f(\rho_\ell) - f(\rho_r)]}{\rho_\ell - \rho_r} \quad (20)$$

A rarefaction develops if  $f'(\rho_\ell) < f'(\rho_r)$ . The rarefaction can be entirely to the left, or to the right or in the middle.

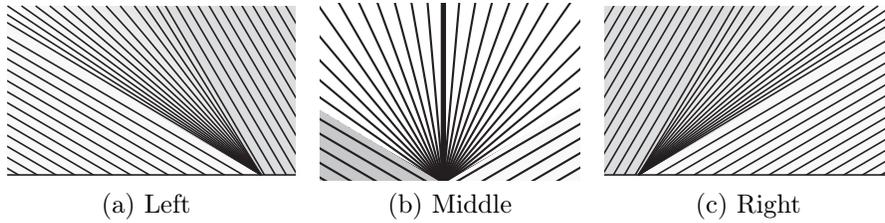


Figure 9: Rarefaction Solution

### 3.1.2 Characteristics for a Traffic Problem

Figure 10 shows a  $x - t$  plot for traffic density  $\rho(t, x)$ . Initially the traffic density is constant at  $\rho_0$ . At time  $t = 0$ , there is a traffic light at  $x = 0$  that turns red. We see the shockwaves travelling backward so that there is a discontinuity between traffic density being  $\rho_0$  to the left of the shock line and being  $\rho_m$  to the right of it. On the right there is another shockwave traveling to the right between zero traffic density and  $\rho_0$ . At time  $t = t_c$ , the light turns green and we see rarefaction of traffic starting at  $x = 0$ . Corresponding to time  $t = t_u$  we see the plot of traffic density  $\rho(t_u, x)$  that shows to the two shock waves as well as rarefaction of the traffic density. This shows that the traffic solution has discontinuities and a weak solution of the LWR model is required that allows for these discontinuous solutions. A basic presentation of traffic dynamics and characteristics are shown in [13], and we strongly advise any reader who does not have the background in mathematical traffic theory to read that.

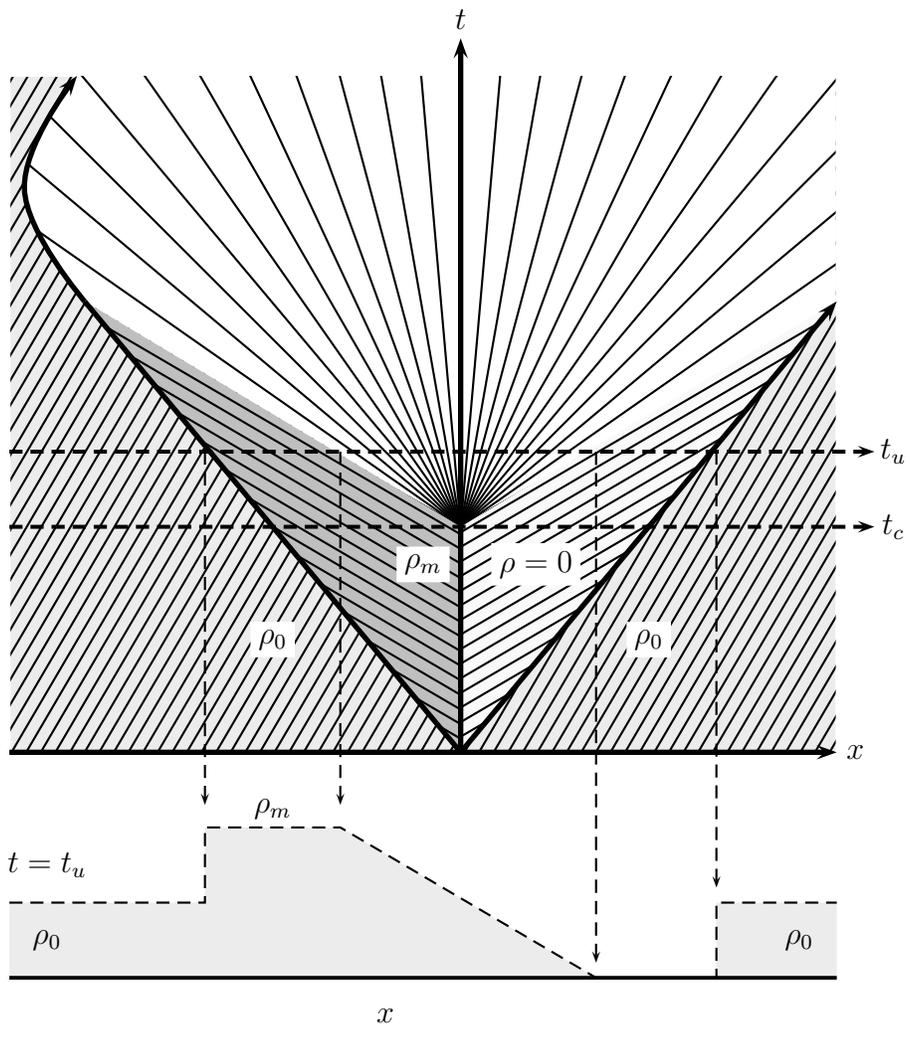


Figure 10: Traffic Characteristics

### 3.1.3 Generalized Solutions

For a conservation law

$$\rho_t + f(\rho)_x = 0 \quad (21)$$

with initial condition

$$\rho(x, 0) = \rho_0(x), \quad (22)$$

where  $\rho_0(x) \in L^1_{loc}(R; R^n)$ , solution in the distributional sense is defined below for smooth vector field  $f : R^n \rightarrow R^n$  (see [5]).

**Definition 3.1.** A measurable locally integrable function  $\rho(t, x)$  is a solution in the distributional sense of the Cauchy problem 21 if for every  $\phi \in C_0^\infty(R^+ \times R) \mapsto R^n$

$$\begin{aligned} \iint_{R^+ \times R} [\rho(t, x) \phi_t(t, x) + f(\rho(t, x)) \phi_x(t, x)] dx dt \\ + \int_R \rho_0(x) \phi(x, 0) dx = 0 \end{aligned} \quad (23)$$

□

### 3.1.4 Weak Solutions

A measurable locally integrable function  $\rho(t, x)$  is a weak distributional solution of the Cauchy problem (21) if it is a distributional solution in  $(0, T) \times \mathbb{R}$ , 22 and if  $\rho$  is continuous as a function from  $[0, T]$  into  $L^1_{loc}$ . We assume  $\rho(t, x) = \rho(t, x^+)$  and the continuity condition implies

$$\lim_{t \rightarrow 0} \int_R |\rho(t, x) - \rho_0(x)| dx = 0 \quad (24)$$

## 3.2 Scalar Initial-Boundary Problem

Consider the scalar conservation law given here.

$$\rho_t + f(t, x, \rho)_x = 0 \quad (25)$$

with initial condition

$$\rho(0, x) = \rho_0(x), \quad (26)$$

and boundary conditions

$$\rho(t, a) = \rho_a(t) \text{ and } \rho(t, b) = \rho_b(t), \quad (27)$$

The boundary conditions cannot be prescribed point-wise, since characteristics from inside the domain might be traveling to outside at the boundary. In that case, the data at the boundary influences the local dynamics at the boundary but does not become equal to the value at the boundary. This is shown in Figure 11 where for some time boundary data on the left can be prescribed when characteristics from the boundary can be *pushed in* (see [35]). However when the characteristics are coming from inside, the boundary data can not be prescribed.

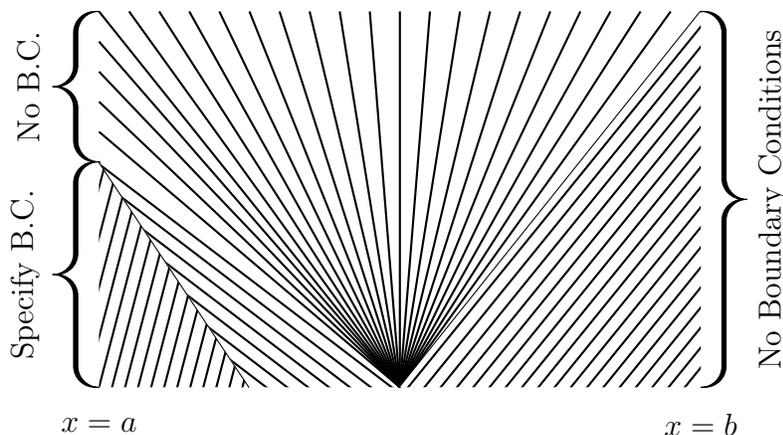


Figure 11: Boundary Data

### 3.3 Derivation of the Travel Time Dynamics

This section provides a model for obtaining the experienced travel time function for the hydrodynamic model, LWR for traffic.

Consider a link as shown in the Figure 12. We want to develop a travel time function  $T(t, x)$  that provides the travel time for a vehicle at position  $x$  and time  $t$  to reach  $x = \ell$ . It takes a vehicle time  $\Delta x/v(t, x)$  to move from  $x$  to  $x + \Delta x$ . Hence, we have the following travel time condition.

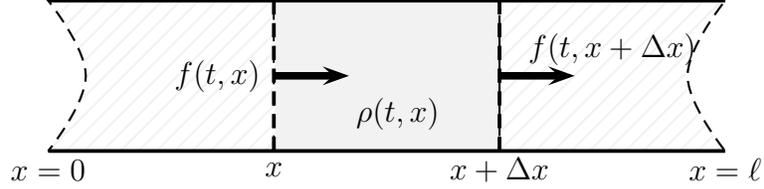


Figure 12: Travel Time on a Link

$$T(t + \Delta t, x + \Delta x) = T(t, x) - \frac{\Delta x}{v(t, x)} \quad (28)$$

Taking the Taylor series first terms for  $T(t, x)$  and simplifying, we obtain

$$\frac{\partial T(t, x)}{\partial t} \Delta t + \frac{\partial T(t, x)}{\partial x} \Delta x = -\frac{\Delta x}{v(t, x)} \quad (29)$$

Multiplying by  $v(t, x)$ , dividing by  $\Delta x$ , and then taking limits and simplifying we get the travel time partial differential equation.

$$\frac{\partial T(t, x)}{\partial t} + \frac{\partial T(t, x)}{\partial x} v(\rho(t, x)) + 1 = 0 \quad (30)$$

Another way to derive this equation is as follows. Figure 13. shows the straight line characteristics for traffic density, on which a vehicle trajectory is superimposed. The vehicles initial position at time 0 is  $x_0$  and the reciprocal of the magnitude of the tangent of the vehicle trajectory curve gives the speed of the vehicle ( $dx/dt$ ). The vehicle trajectory curve  $X(t)$  provides the vehicle characteristic curves as compared to the traffic density characteristics. On this trajectory the travel time to  $x = \ell$  decreases at the same rate as time, and hence, we have

$$\frac{dT(t, x(t))}{dt} = -1 \quad (31)$$

On the trajectory  $x(t)$ , we apply the chain rule to  $T(t, x(t))$  to obtain

$$\frac{\partial T(t, x)}{\partial t} + \frac{\partial T(t, x)}{\partial x} \frac{dx}{dt} + 1 = 0 \quad (32)$$

Since  $dx/dt$  is the vehicle speed  $v(t, x)$ , we obtain the equation 30.

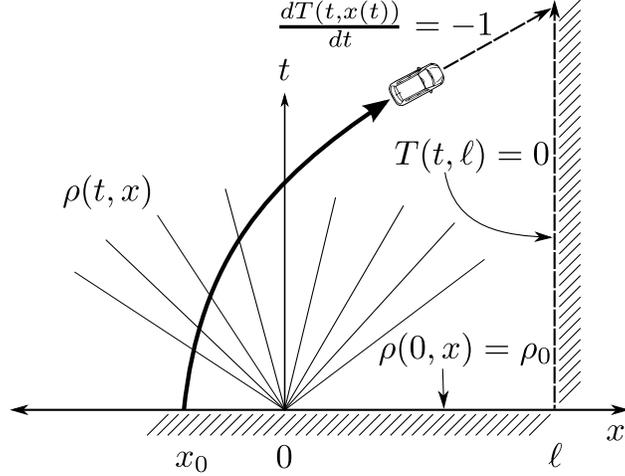


Figure 13: Vehicle Trajectory

### 3.4 Wellposedness and Solution Properties

The one-way coupled PDE system for LWR and travel time for a link is given by

$$\begin{aligned} \frac{\partial}{\partial t} \rho(t, x) + \frac{\partial}{\partial x} [\rho(t, x) v(\rho(t, x))] &= 0 \\ \frac{\partial T(t, x)}{\partial t} + \frac{\partial T(t, x)}{\partial x} v(\rho(t, x)) + 1 &= 0 \\ v(\rho(t, x)) &= v_f \left(1 - \frac{\rho}{\rho_m}\right) \end{aligned} \quad (33)$$

We also have the initial condition on the traffic dynamics as  $\rho(0, x) = \rho_0$  and a one sided boundary condition on the travel time dynamics as  $T(t, \ell) = 0$ . In fact, we have  $\forall x \geq 0 \ T(t, \ell) = 0$ .

The strategy for proving the existence of the solution to the mixed boundary value asymmetrically coupled hyperbolic partial differential system given by Equation 33 is based on the technique of wave front tracking. The existence proof of the traffic density equation follows directly the steps for the scalar conservation laws as shown in [5] or [14]. However, the proof for the travel time equation is being constructed here for the first time.

For the traffic density equation, we will show that it satisfies the entropy kruzkov solution ([23]).

**Definition 3.2** (Kruzkov Solution). The Kruzkov entropy solution is any function  $\rho : [0, \infty) \rightarrow L_{\text{loc}}^{\mathbb{R}}$ , such that  $\forall k > 0, \phi > 0 \in C_c^1(\mathbb{R}^2)$  with compact support of  $\phi$  being in  $t > 0$ , we have

$$\iint [|\rho - k|\phi_t + (f(\rho) - f(k)) \text{sgn}(\rho - k)\phi_x] dx dt \geq 0 \quad (34)$$

and there exists a set  $E$  of zero measure on  $[0, T]$ , such that for  $t \in [0, T] \setminus E$ , the function  $\rho(t, x)$  is defined almost everywhere in  $\mathbb{R}$ , and for any ball  $K_r = \{|x| \leq r\}$

$$\lim_{t \rightarrow 0} \int_{K_r} |\rho(t, x) - \rho_0(x)| dx = 0. \quad (35)$$

□

Inequality 34 is equivalent to condition  $E$  in [29], if  $(\rho_-, \rho_+)$  is a discontinuity of  $\rho$  and  $v$  is any number between  $\rho_-$  and  $\rho_+$ , then

$$\frac{f(x, t, v) - f(x, t, \rho_-)}{v - \rho_-} \geq \frac{f(x, t, \rho_+) - f(x, t, \rho_-)}{\rho_+ - \rho_-} \quad (36)$$

The Kruzkov condition comes from using the following entropy flux pair.

$$\eta(u) = |u - k| \quad \text{and} \quad q(u) = \text{sgn}(u - k) \cdot (f(u) - f(k)) \quad (37)$$

The proof will be based on building a sequence of piecewise constant solutions to the traffic density conservation law, and building a corresponding consistent sequence of travel time function solutions. The steps to build the density solution will follow closely the steps given in [5], and then the

corresponding travel time solutions will be developed in parallel to those steps in order to develop the complete solution. In order to extract the solution to the model in equation 33, we will need the following intermediate results.

Generalized solutions have a nice convergence property that is stated and proved here (see [5]).

**Lemma 3.3.** *If  $u_n$  is a sequence of distributional solutions to the conservation law (21), then*

1.  $(u_n \rightarrow u, f(u_n) \rightarrow f(u) \text{ in } L^1_{loc}) \Rightarrow u$  is a solution of the conservation law (21).
2.  $(u_n \rightarrow u \text{ in } L^1_{loc} \text{ and if all } u_n \text{ take values in a compact set}) \Rightarrow u$  is a solution of the conservation law (21).

□

*Proof.* 1. Assume that  $u_n \rightarrow u$ , and  $f(u_n) \rightarrow f(u)$  in  $L^1_{loc}$  and that  $\phi \in C^1_0$  then we obtain

$$\begin{aligned}
& \left| \iint_{\Omega} \{u_n \phi_t + f(u_n) \phi_x\} dt dx - \iint_{\Omega} \{u \phi_t + f(u) \phi_x\} dt dx \right| \\
& \leq \iint_{\Omega} \{|u_n - u| |\phi_t| + |f(u_n) - f(u)| |\phi_x|\} dt dx \\
& \leq \|u_n - u\|_{supp\phi,1} \|\phi_t\|_{\infty} + \|f(u_n) - f(u)\|_{supp\phi,1} \|\phi_x\|_{\infty} \\
& \rightarrow 0, \text{ as } n \rightarrow \infty \quad (38)
\end{aligned}$$

2. If we assume that  $u_n$  take values in a fixed compact subset  $K$  of  $\Omega$  and show that  $f(u_n) \rightarrow f(u)$  in  $L^1_{loc}$ , then we can show that the first part implies the second part. As  $f$  is a smooth vector field,  $f$  is uniformly bounded on compact subsets. As the values of  $u_n$  stay inside  $K$ , it follows that  $f(u_n)$  is uniformly bounded, say  $\|f(u_n)\| \leq M$ . Then  $\|f(u_n(x)) - f(u(x))\| \leq 2M$  for all  $x$  in the support of  $\phi$  where the constant function  $2M$  is integrable over the support of  $\phi$ .

Since  $u_n \rightarrow u$  in  $L^1_{loc}$  in a subsequence, we can also assume that  $u_n \rightarrow u$  pointwise on the support of  $\phi$  and hence also  $f(u_n) \rightarrow f(u)$  on the support of  $\phi$ . We now can use the *Lebesgue Dominated Convergence*

*Theorem* (see [32] or [10]) to see that  $f(u_n) \rightarrow f(u)$  in  $L^1_{\text{supp}\phi}$  (or, more generally, in  $L^1_{\text{loc}}$ ) as required.  $\square$

One very important theorem that we need for convergence deals with the sequential compactness property of sequence of functions of bounded variations (*BV*).

**Theorem 3.4. (*Helly*)** Consider a sequence of functions given by  $f_n : \mathbb{R} \rightarrow \mathbb{R}^n$  such that

$$\text{Total Variation}\{f_n\} \leq C_1, \quad |f_n(x)| \leq C_2 \text{ for all } n, x \quad (39)$$

for constants  $C_1, C_2$ . Then, there exists a function  $f$  and a subsequence  $f_{n_k}$  such that

$$\begin{aligned} \lim_{n_k \rightarrow \infty} f_{n_k}(x) &= f(x) \text{ for every } x \in \mathbb{R} \\ \text{Total Variation}\{f\} &\leq C_1, \quad |f(x)| \leq C_2 \text{ for all } x \end{aligned} \quad (40)$$

$\square$

The following lemma is taken from [5].

**Lemma 3.5** (Piecewise Constant approximation of BV Functions). For any given function of bounded variation (*BV*),  $\rho : \mathbb{R} \rightarrow \mathbb{R}^n$  which is right continuous,  $\forall \epsilon > 0, \exists \phi$ , a piecewise constant function, such that

$$\begin{aligned} \text{Total Variation}\{\phi\} &\leq \text{Total Variation}\{\rho\} \\ \|\rho - \phi\|_{L^\infty} &\leq \epsilon \end{aligned} \quad (41)$$

moreover,

$$\begin{aligned} \int_{-\infty}^0 |\rho(x) - \rho(-\infty)| dx + \int_0^\infty |\rho(x) - \rho(\infty)| dx < \infty \\ \Rightarrow \|\rho - \phi\|_{L_1} < \epsilon \end{aligned} \quad (42)$$

$\square$

Finally, we present the most important theorem for our existence proof whose proof can be found in [5].

**Theorem 3.6** (Subsequence Convergence for BV in Space and Lipschitz in Time). *Given a sequence of functions  $\rho_i : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}^n$ , such that*

$$\begin{aligned} \forall t, x, \text{ Total Variation } \{\rho_i(t, \cdot)\} &\leq C_1, |\rho_i(t, x)| \leq C_2 \\ \forall t_1, t_2 \geq 0 \int_{-\infty}^{\infty} |\rho_i(t_1, x) - \rho_i(t_2, x)| dx &\leq L|t_1 - t_2| \\ \Rightarrow \exists \rho \in BV \cap L_{loc}^1 \end{aligned} \quad (43)$$

and a subsequence  $\rho_{i_k}$  such that

$$\begin{aligned} \|\rho - \rho_{i_k}\|_{L_{loc}^1} &\rightarrow 0, \\ \forall t, x, \text{ Total Variation } \{\rho(t, \cdot)\} &\leq C_1, |\rho(t, x)| \leq C_2, \text{ and} \\ \forall t_1, t_2 \geq 0 \int_{-\infty}^{\infty} |\rho(t_1, x) - \rho(t_2, x)| dx &\leq L|t_1 - t_2| \end{aligned} \quad (44)$$

where we take point values of  $\rho$  to keep it right continuous.  $\square$

### 3.4.1 Piecewise Constant Approximation with Travel Time Field

To obtain a sequence of approximate solutions we perform the following steps starting with  $i = 0$ .

1. Divide the interval  $[0, \rho_m]$  into  $2^i$  intervals of equal length. For instance, when  $i = 1$ , we will have two intervals,  $[0, \rho_m/2]$  and  $[\rho_m/2, \rho_m]$ .
2. Approximate the initial density  $\rho_0$  by a piecewise constant function  $\rho_{0_i}$  that takes only the values of the end points of the intervals in the previous step. See figure 14d, where different approximations for a given function (BV in general, although in this case smooth) are given for various number of discrete values possible from the intervals.
3. Develop a piecewise linear approximation of the flow function  $f(\rho)$  which uses values of the function only at the same discrete values of  $\rho$  for the discretization  $i$  and joining them with straight lines to create the function  $f_i(\rho)$ . See Figure 15 to see an example of a piecewise linear function from some discrete  $\rho$  values on the  $x$ -axis.
4. Using Riemann solutions, and then using characteristics solve the following mixed initial boundary value problem, where we have used

$$f(\rho_i) = \rho_i(t, x)v(\rho_i(t, x)).$$

$$\begin{aligned} \frac{\partial}{\partial t}\rho_i(t, x) + \frac{\partial}{\partial x}[f_i(\rho_i)] &= 0 \\ \frac{\partial T_i(t, x)}{\partial t} + \frac{\partial T_i(t, x)}{\partial x}v(\rho_i(t, x)) + 1 &= 0 \\ v(\rho_i(t, x)) &= v_f\left(1 - \frac{\rho_i}{\rho_m}\right) \end{aligned} \quad (45)$$

with

$$\begin{aligned} \rho_i(0, x) &= \rho_{0i} \\ T_i(t, \ell) &= 0 \end{aligned} \quad (46)$$

We will discuss the existence of the solution to this approximate problem below.

5. Update  $i$  to  $i + 1$ , and then iterate through these steps to generate a sequence of approximate solutions  $\rho_i$  and  $T_i$ .
6. Prove the convergence of some subsequences and use them to develop the solution to the original problem with the correct boundary condition for the travel time function.

Now, we will present the details of these steps. We will assume that  $\forall x \rho_0(x) \in [0, \rho_M]$  where  $\rho_M < \rho_m$ . Now, first we will give the solution to the Riemann problems for these approximations.

### Solution to Riemann Problems for the Piecewise Constant Approximation

**Case 1** ( $\rho_\ell < \rho_r$ ): We define a function  $f_\ell$  to be the largest convex less than  $f$  on  $x \in [\rho_\ell, \rho_r]$ . This case is shown in the left subfigure in Figure 15. This creates a shock wave dictated by the slope of  $f_\ell$  between the two points. Hence, the shock is either as shown in Figure 8a or the one shown in Figure 8b.

**Case 2** ( $\rho_r < \rho_\ell$ ): We define a function  $f_h$  to be the smallest concave greater than  $f$  on  $x \in [\rho_r, \rho_\ell]$ . This case is shown in the right subfigure in Figure 15. This creates a rarefaction dictated by the range of slopes of  $f$  at the end points. Hence, the rarefaction is either as shown in Figure 9a or the one shown in Figure 9b or the one shown in Figure 9c.

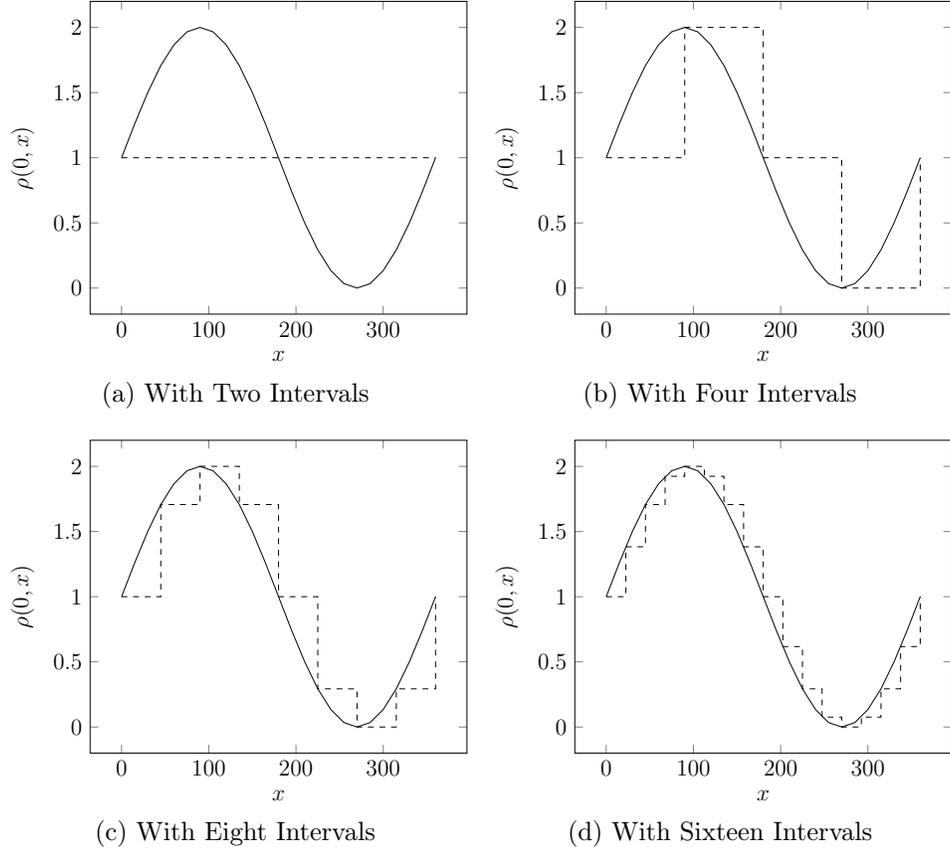


Figure 14: Piecewise Constant Approximation

Now, let's apply these two cases to  $f_i$  instead of  $f$ . In case 1, there is a single shock, and in case 2, there will be as many shocks (instead of rarefaction, as the rarefaction becomes piecewise constant) as there are vertices on the piecewise linear function  $f_i^h$ , where  $f_i^h$  is the smallest concave greater than  $f_i$  on  $x \in [\rho_r, \rho_\ell]$  as shown in the right subfigure in Figure 15. Precisely for the left subfigure of Figure 15, there will be a shock with negative slope, and for the right subfigure, there will be three shocks, two with negative slopes, and one positive. The densities at  $t^-$  time were  $\rho_\ell$  and  $\rho_r$ , but at time  $t^+$  there will be two more densities corresponding to the two vertices, which we show as  $\rho_1$  and  $\rho_2$  in Figure 16.

Now, we show that in both cases, case 1 and case 2, the constructed solution to the Riemann problem satisfies the Kruzkov conditions given by

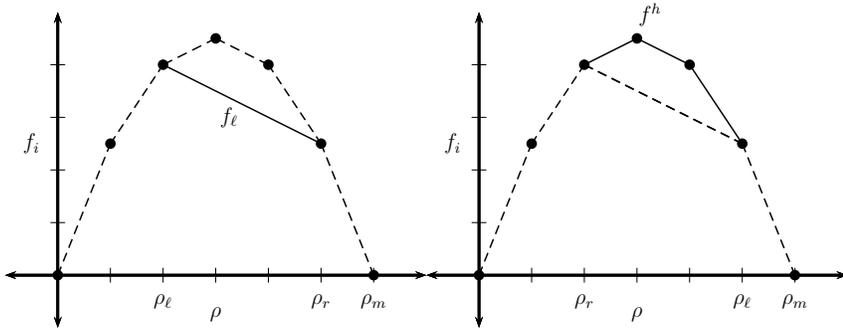


Figure 15: Piecewise Linear Flow

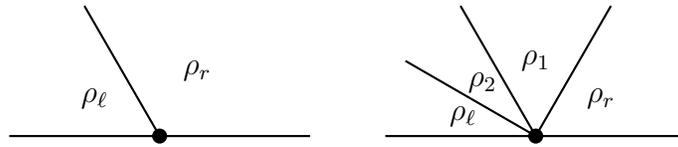


Figure 16: Shocks for the Piecewise Linear Example

Equation 34 and Equation 35.

We need to show that

$$\iint [[\rho_{iR} - k] \phi_t + (f_i(\rho_{iR}) - f_i(k)) \operatorname{sgn}(\rho_{iR} - k) \phi_x] dx dt \geq 0 \quad (47)$$

and

$$\lim_{t \rightarrow 0} \int_{K_r} |\rho_{iR}(t, x) - \rho_{iR0}(x)| dx = 0. \quad (48)$$

Notice that since we are solving the Riemann problem using the approximated densities, and not the actual one with the approximated initial conditions, we refer to this solution and the initial conditions using the subscript  $iR$  as used in Equation 47 and Equation 48. For instance, for the Riemann problem,  $\forall x \leq 0, \rho_{iR0}(x) = \rho_\ell$ , and  $\forall x > 0, \rho_{iR0}(x) = \rho_r$ , where  $\rho_\ell$  and  $\rho_r$  have to belong to the finitely many values allowed for the discretization level

$i$ , and the corresponding  $f_i$  is a piecewise linear function built out of values of  $f$  at those discretization density points.

**Proof for Case 1** ( $\rho_\ell < \rho_r$ ): The solution is precisely give by:

$$\rho_{iR}(t, x) = \begin{cases} \rho_\ell, & \text{if } \frac{x}{t} < \lambda \\ \rho_r, & \text{otherwise} \end{cases} \quad (49)$$

where  $\lambda$  is the shock speed give by

$$\lambda = \frac{f_i(\rho_r) - f_i(\rho_\ell)}{\rho_r - \rho_\ell} \quad (50)$$

To verify that this is the Kruzkov entropy solution, we perform the following steps.

$$\begin{aligned} & \iint [|\rho_{iR} - k|\phi_t + (f_i(\rho_{iR}) - f_i(k)) \operatorname{sgn}(\rho_{iR} - k)] dx dt \\ &= \int [ (|\rho_r - k|\lambda - (f(\rho_r) - f(k)) \operatorname{sgn}(\rho_r - k)) \\ & \quad - (|\rho_\ell - k|\lambda - (f(\rho_\ell) - f(k)) \operatorname{sgn}(\rho_r - k))] \phi(t, \lambda t) dt \\ &= \int [(\rho_\ell + \rho_r - 2k)\lambda + 2f_i(k) - f_i(\rho_\ell) - f_i(\rho_r)] \chi_{[\rho_\ell, \rho_r]} \phi(t, \lambda t) dt \geq 0 \end{aligned} \quad (51)$$

The last step in the proof comes from the fact that

$$f_i(k) \geq f_\ell(k) = \frac{1}{2} \{ [f_i(\rho_\ell) + (k - \rho_\ell)\lambda] + [f_i(\rho_r) + (k - \rho_r)\lambda] \} \quad (52)$$

Now we need to find the solution for the travel time equation using this solution of the density. We now need to solve the following semilinear PDE.

$$\begin{aligned} \frac{\partial T_{iR}(t, x)}{\partial t} + \frac{\partial T_{iR}(t, x)}{\partial x} v(\rho_{iR}(t, x)) + 1 &= 0 \\ v(\rho_{iR}(t, x)) &= v_f(1 - \frac{\rho_{iR}}{\rho_m}) \end{aligned} \quad (53)$$

We now need to solve the characteristics for this semilinear PDE. Notice that  $v(\rho_{iR}(t, x))$  is not Lipschitz in terms of  $t$  and  $x$ . In fact, we know that  $\rho_{iR}$  has discontinuity. We can still build characteristics on the two sides of the

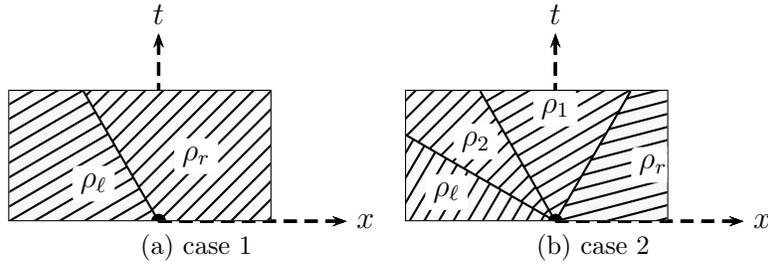


Figure 17: Vehicle Characteristics

discontinuity, and then look for Caratheodory solutions to the characteristic ODEs. A solution such that its characteristics satisfy the ODEs almost everywhere is called a **broad solution**.

To obtain the solution and properties of the  $T(t, x)$  given  $\rho_i(t, x)$ , we will develop the travel time function for a compact set  $[0, t_f] \times [a, \ell]$ ,  $a < \ell$ . Figure 17a shows the characteristics of vehicles for case 1. The slope of the characteristics show that vehicles travel faster in  $\rho_\ell$  as compared to when they are in  $\rho_r$ .

**Theorem 3.7.** *Given a shock solution to the Riemann problem, the travel time dynamics has a continuous solution of bounded variation in any set  $[0, t_f] \times [a, \ell]$ , where  $a < \ell$ , and for boundary condition  $T(t, \ell) = 0$ .  $\square$*

*Proof.* The value  $T(t, x)$  is computed from characteristics as follows.

$$T(t, x) = \begin{cases} \frac{\ell - x}{v(\rho_r)}, & \text{if } \frac{x}{t} > \lambda \\ \frac{x - \lambda t}{\lambda - v(\rho_\ell)} + \frac{\ell - \lambda t_1}{v(\rho_r)}, & \text{otherwise} \end{cases} \quad (54)$$

where

$$t_1 = \frac{x - v(\rho_\ell)t}{\lambda - v(\rho_\ell)}$$

The derivation of this formula for  $(x/t) \leq \lambda$  can be easily derived by considering Figure 18, and noticing that the slope of the trajectory to the left of the shock line is dictated by  $v(\rho_\ell)$  and the slope of the trajectory to the right of the shock line is dictated by  $v(\rho_r)$ .

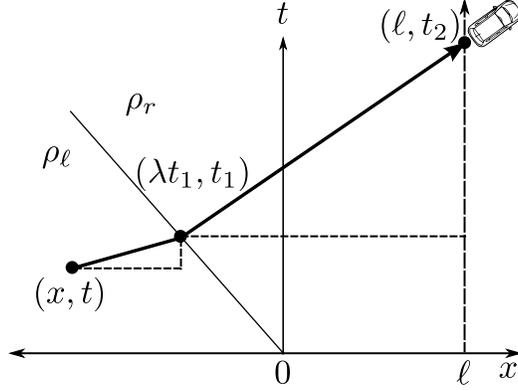


Figure 18: Travel Time from Vehicle Characteristics

This function is clearly a continuous function. Moreover, it is a monotone function in  $t$  as well as  $x$ . Hence, it is a function of bounded variation in the compact interval of  $(t, x)$ .  $\square$

**Proof for Case 2** ( $\rho_r \leq \rho_\ell$ ): The function  $f^h$  in this case goes through some vertices at densities  $\rho_1, \rho_2$ , etc. as can be seen in the right subfigure in Figure 15, also in the right subfigure in Figure 16, and Figure 17b. So, in general, let's rename them as  $u_0 = \rho_\ell > u_1 > u_2 > \dots > u_p = \rho_r$ , and define shock speeds as

$$\lambda_j = \frac{f_i(u_j) - f_i(u_{j-1})}{u_j - u_{j-1}} \quad j = 1, 2, \dots, p \quad (55)$$

Now, the solution is precisely give by:

$$\rho_{iR}(t, x) = \begin{cases} \rho_\ell, & \text{if } \frac{x}{t} < \lambda_1 \\ u_j, & \text{if } \lambda_j < \frac{x}{t} < \lambda_{j+1} \\ \rho_r, & \text{if } \lambda_p < \frac{x}{t} \end{cases} \quad (56)$$

To verify that this is the Kruzkov entropy solution, we perform the following steps.

$$\begin{aligned}
& \iint [|\rho_{iR} - k|\phi_t + (f_i(\rho_{iR}) - f_i(k)) \operatorname{sgn}(\rho_{iR} - k)] dx dt \\
&= \sum_{j=1}^p \int [ (|u_j - k|\lambda - (f(u_j) - f(k)) \operatorname{sgn}(u_j - k)) \\
&\quad - (|u_{j-1} - k|\lambda - (f(u_{j-1}) - f(k)) \operatorname{sgn}(u_{j-1} - k))] \phi(t, \lambda t) dt \\
&= \sum_{j=1}^p \int [(u_{j-1} + u_j - 2k)\lambda + \\
&\quad 2f_i(k) - f_i(u_{j-1}) - f_i(u_j)] \chi_{[u_j, u_{j-1}]} \phi(t, \lambda t) dt \geq 0 \quad (57)
\end{aligned}$$

**Theorem 3.8.** *Given a piecewise shock solution (discretized rarefaction) to the Riemann problem, the travel time dynamics has a continuous solution of bounded variation in any set  $[0, t_f] \times [a, \ell]$ , where  $a < \ell$ , and for boundary condition  $T(t, \ell) = 0$ .  $\square$*

*Proof.* The value  $T(t, x)$  is computed from characteristics as follows.

$$T(t, x) = \begin{cases} \frac{\ell - x}{v(\rho_r)}, & \text{if } \frac{x}{t} > \lambda_p \\ \frac{x - \lambda_j t}{\lambda_j - v(\rho_\ell)} + \sum_{i=j+1}^{p-1} \frac{\lambda_{i+1} t_{i+1} - \lambda_i t_i}{v(u_i)} + \frac{\ell - \lambda t_p}{v(\rho_r)}, & \text{if } \lambda_j > \frac{x}{t} > \lambda_{j-1} \end{cases} \quad (58)$$

where

$$t_{i+1} = \frac{\lambda_i t_i - v(u_i) t_i}{\lambda_{i+1} - v(u_i)}$$

The function is continuous in  $(t, x)$ , monotone in  $t$  and  $x$ , and hence of bounded variation.  $\square$

Now, we study the case where the initial density is discretized and we can use the Riemann solutions to get the solution for this problem. We obtain a solution for this iteration as  $\rho_i(t, x)$  by solving the Riemann problems at every junction. As the wave front move forward in time, in finite time multiple waves can interact. When they interact, two possibilities are there. One is that all the jumps will have the same sign, and their interaction will keep the variation the same and reduce the number of jumps after interaction. If

there is a change in the sign of the jump, then there will be a cancellation effect and the variation will decrease.

The travel time function over the same compact rectangle is continuous and of bounded variation for each iteration. Moreover the travel time solutions for each iteration are also uniformly bounded and have uniform bounded variation. This is proved by considering the travel time when the density in the rectangle is equal to  $\rho_M$ . The variation and the travel time for this case provides the uniform bounds for both.

**Theorem 3.9.** *The one-way coupled travel time model given by Equation 33 has an entropy weak solution  $\rho(t, x)$  for the traffic density part of the equation defined for all  $t \geq 0$  and a broad solution for the travel time field which is of bounded variation on any compact interval for  $t \geq 0$  when the initial density  $\rho_0 \in L^1$  is of bounded variation. Moreover, we have the following properties of the solution.*

$$\begin{aligned}
& \text{Total Variation } \{\rho(t, \cdot)\} \leq \text{Total Variation } \{\rho_0\}, \quad \forall t \geq 0, \\
& \|\rho(t, \cdot)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} \quad \forall t \geq 0, \\
& t_1 \geq t_2 \Rightarrow T(t_1, x) \geq T(t_2, x), \quad \text{and} \\
& x_1 \geq x_2 \Rightarrow T(t, x_1) \leq T(t, x_2)
\end{aligned} \tag{59}$$

□

*Proof.* From our initial conditions of each iteration we have the following properties:

1.  $\|\rho_{i_0}\|_{L^\infty} \leq \rho_M$
2.  $\text{Total Variation } \{\rho_{i_0}\} \leq \text{Total Variation } \{\rho_0\}$
3.  $\|\rho_{i_0} - \rho_0\|_{L^1} \rightarrow 0$

For each approximated initial condition  $\rho_{i_0}$ , we have obtained a piecewise constant density  $\rho_i(t, x)$  and a corresponding monotone  $T_i(t, x)$ , such that

$$\begin{aligned}
& \text{Total Variation } \{\rho_i(t, \cdot)\} \leq \text{Total Variation } \{\rho_0\}, \quad \text{and } |\rho_i(t, x)| \leq \rho_M, \\
& |f(\rho_1) - f(\rho_2)| \leq v_f |\rho_1 - \rho_2| \quad \forall \rho_1, \rho_2 \in [0, \rho_M], \\
& \text{Total Variation } \{T_i\} \leq K_1, \quad \text{and} \\
& T_i(t, x) \leq K_2, \quad \forall (t, x) \in \Omega
\end{aligned} \tag{60}$$

Here,  $K_1$  and  $K_2$  are constant obtained for the compact rectangle  $\Omega$  in  $t \geq 0$ .

From these we obtain,

$$\|\rho_i(t_1, \cdot) - \rho_i(t_2, \cdot)\|_{L^1} \leq v_f |t_1 - t_2| \text{Total Variation } \{\rho_0\} \quad (61)$$

Now, we can apply the Theorem 3.6 to obtain a subsequence of  $\rho_i$  converging in  $L_1$  to  $\rho$ . Since  $f_i \rightarrow f$  uniformly on  $[0, \rho_M]$ , lemma 3.3 shows that the traffic density solution is the Kruzkov entropy solution, as shown here.

$$\begin{aligned} & \iint [|\rho - k| \phi_t + (f(\rho) - f(k)) \text{sgn}(\rho - k) \phi_x] dx dt \\ &= \lim_{i \rightarrow \infty} \iint [|\rho_i - k| \phi_t + (f(\rho_i) - f(k)) \text{sgn}(\rho - k) \phi_x] dx dt \geq 0 \end{aligned} \quad (62)$$

since each iterate approximate solution is a Kruzkov weak entropy solution. The  $L_1$  convergence of the initial conditions to the actual initial condition combined with Equation 61 proves that the solution converges in  $L_1$  to its initial given data.

Now, from the uniformly bounded sequence  $T_i(\cdot, \cdot)$  which also has uniform bounded variation, we obtain a convergent subsequence with limit  $T(\cdot, \cdot)$  which has a weak\* convergent subsequence. To establish broad solutions, consider the following equation, where  $x_i(t)$  give the characteristic curves over each iterate approximate solution  $\rho_i(t, x)$ .

$$x_i(t) = x(0) + \int_0^t v(\rho_i(\tau, x(\tau))) d\tau \quad (63)$$

Since,  $0 \leq v(\rho) \leq v_f$  is a continuous function of  $\rho$ , we apply Lebesgue's dominated convergence theorem to obtain our result.

$$\begin{aligned} x(t) &= \lim_{i \rightarrow \infty} x_i(t) = x(0) + \lim_{i \rightarrow \infty} \int_0^t v(\rho_i(\tau, x(\tau))) d\tau \\ &= \int_0^t \lim_{i \rightarrow \infty} v(\rho_i(\tau, x(\tau))) d\tau = \int_0^t v(\rho(\tau, x(\tau))) d\tau \end{aligned} \quad (64)$$

□

**Theorem 3.10.** *Travel time function  $T(t, x)$  restricted to the set where it converges from the sequence  $T_i$  is everywhere continuous function of  $(t, x) - \Omega$ , where the subtraction symbol signifies set difference and  $\Omega$  here is the set where the convergence does not take place.  $\square$*

*Proof.* We know that each function  $T_i$  is continuous, and also the convergence is everywhere in  $(t, x) - \Omega$ . Hence, we get

$$|T(t_2, x_2) - T(t_1, x_1)| \leq |T(t_2, x_2) - T_j(t_2, x_2)| + |T_j(t_2, x_2) - T_j(t_1, x_1)| + |T_j(t_1, x_1) - T(t_1, x_1)| \rightarrow 0 \quad (65)$$

$\square$

**Theorem 3.11.** *The sequence of travel time functions  $T_i$  has a subsequence that converges on a dense subset of  $(t, x)$  to  $T(t, x)$ .  $\square$*

*Proof.* Since each  $T_i$  is uniformly bounded, we take all the points with rational coordinates, a set dense in  $(t, x)$  and apply diagonalization to obtain a subsequence that converges at all those points.  $\square$

## 4 Conclusions

This paper presented a traffic density based travel time function as compared to flow based one. This removed an inherent difficulty with flow based travel time functions. The paper then developed the temporal and spacial dynamics of the travel time function and proved the existence of the solution of the LWR traffic model coupled with the travel time dynamics.

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