CVL 851: Assignment #1

Special Topics in Transportation

Due on Monday, February 27^{th}, 2023

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Contents

Problem 1. *State Peano's axioms. (**Note**: Use wikipedia or any textbook, or any other resource for proper axiomatic definitions)*

Solution:

The five Peano axioms are:

1. Zero is a natural number.
   $0 \in \mathbb{N}$
2. Every natural number has a successor in the natural numbers.
   $\forall n \in \mathbb{N},\ S(n) \in \mathbb{N}$.
3. Zero is not the successor of any natural number.
   $\nexists\ n \in \mathbb{N}, S(n) = 0$
4. If the successor of two natural numbers is the same, then the two original numbers are the same.
   $\forall n \in \mathbb{N}, \forall m \in \mathbb{N}, S(n) = S(m) \implies n = m$
5. (**Induction**): If a set contains zero and the successor of every number is in the set, then the set contains the natural numbers.
   If $0 \in \Omega$ and $n \in \Omega \implies S(n) \in \Omega$, then $\Omega \subset \mathbb{N}$

Problem 2. 1. *Prove that the cardinality of the set of natural numbers* $\mathbb{N}$, *i.e.* $\{0, 1, 2, \cdots, \}$ *and the set of positive integers* $\mathbb{Z}^+$, *i.e.* $\{1, 2, \cdots, \}$ *is the same.*

2. *Give a set whose cardinality is higher than the cardinality of the set of real numbers* $\mathbb{R}$.

Solution:

1. *Proof.* The bijection $f : \mathbb{N} \to \mathbb{Z}^+$ given by $f(n) = n+1$ proves the same cardinality of the two sets. $\square$

2. The cardinality of the power set of $\mathbb{R}$, given by $2^{\mathbb{R}}$ has a higher cardinality than $\mathbb{R}$ because of Russel's theorem.

Problem 3. *Prove that the set of rational numbers $\mathbb{Q}$ is countable.*

Solution:

Proof. Consider the following sequence of rational numbers.

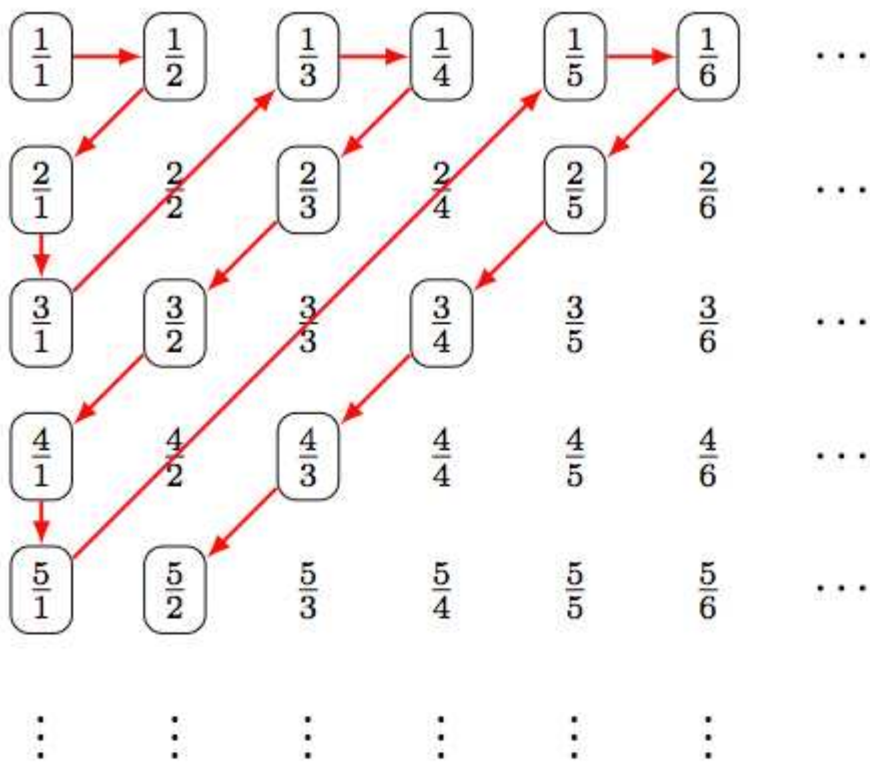

Image source: Quora comment by Lukas Schmidinger
`https://www.quora.com/How-do-you-prove-that-the-set-of-rational-numbers-is-countable`

In the figure, the rational numbers that are the same such as 1/1 and 2/2 are counted only once. The sequence shows the desired bijection.

$\square$

Problem 4. *Prove that the interval* $[0,1] \in \mathbb{R}$ *is uncountable.*

Solution:

Proof. Consider that any sequence of real numbers from $[0,1]$ where we have used only unique representations for each real number in the set is given.

$$0.p_{11}p_{12}p_{13}p_{14}p_{15}\cdots$$
$$0.p_{11}p_{12}p_{13}p_{14}p_{15}\cdots$$
$$0.p_{11}p_{12}p_{13}p_{14}p_{15}\cdots$$

The following real number in $[0,1]$ defined as follows is not in this sequence, proving that any given sequence can not enumerate all numbers in the interval $[0,1] \in \mathbb{R}$, hence that set is uncountable.

$$0.q_1q_2q_3q_4\cdots$$

where

$$q_i = \begin{cases} 0 & \text{if } p_{ii} \text{ odd} \\ 1 & \text{otherwise} \end{cases}$$

□

Problem 5. *Define a groupoid, a semigroup, a monoid, a group, a ring, a field, and a vector space.*

Solution:

Groupoid A groupoid is a tuple $\{\Omega, \odot\}$ containing set Ω and binary operator that is closed in the groupoid. Specifically,

- $p \in \Omega, q \in \Omega \implies p \odot q \in \Omega$ [Closure Property]

Semigroup A semigroup is a grouoid $\{\Omega, \odot\}$ with $\odot$ being associative. Specifically,

- $p \in \Omega, q \in \Omega \implies p \odot q \in \Omega$ [Closure Property]
- $p \in \Omega, q \in \Omega, r \in \Omega \implies (p \odot q) \odot r = p \odot (q \odot r)$ [Associativity Property]

Monoid A monoid is a semigroup $\{\Omega, \odot\}$ containing an identity element. Specifically,

- $p \in \Omega, q \in \Omega \implies p \odot q \in \Omega$ [Closure Property]
- $p \in \Omega, q \in \Omega, r \in \Omega \implies (p \odot q) \odot r = p \odot (q \odot r)$ [Associativity Property]
- $\exists e \in \Omega, \forall p \in \Omega, p \odot e = e \odot p = p$ [Existence of Identity]

Group A group is a monoid $\{\Omega, \odot\}$ containing inverses of each element. Specifically,

- $p \in \Omega, q \in \Omega \implies p \odot q \in \Omega$ [Closure Property]
- $p \in \Omega, q \in \Omega, r \in \Omega \implies (p \odot q) \odot r = p \odot (q \odot r)$ [Associativity Property]
- $\exists e \in \Omega, \forall p \in \Omega, p \odot e = e \odot p = p$ [Existence of Identity]
- $\forall p \in \Omega, \exists p^{-1} \in \Omega, p \odot p^{-1} = p^{-1} \odot p = e$ [Existence of Inverses]

Ring A ring is a triple $\{\Omega, \oplus, \odot\}$ such that $\{\Omega, \oplus\}$ is an abelian (also called commutative) group, i.e. satisfies

- $p \in \Omega, q \in \Omega \implies p \oplus q \in \Omega$ [Closure Property]
- $p \in \Omega, q \in \Omega, r \in \Omega \implies (p \oplus q) \oplus r = p \oplus (q \oplus r)$ [Associativity Property]
- $\exists 0 \in \Omega, \forall p \in \Omega, p \oplus 0 = 0 \oplus p = p$ [Existence of Identity]
- $\forall p \in \Omega, \exists -p \in \Omega, p \oplus -p = -p \oplus p = 0$ [Existence of Inverses]
- $p \in \Omega, q \in \Omega, \implies p \oplus q = q \oplus p$ [Abelian/Commutativity Property]

and $\{\Omega, \odot\}$ is a monoid,

- $p \in \Omega, q \in \Omega \implies p \odot q \in \Omega$ [Closure Property]
- $p \in \Omega, q \in \Omega, r \in \Omega \implies (p \odot q) \odot r = p \odot (q \odot r)$ [Associativity Property]
- $\exists 1 \in \Omega, \forall p \in \Omega, p \odot 1 = 1 \odot p = p$ [Existence of Identity]

satisfying the following two distributive laws connecting operator $\odot$ with the operator $\oplus$.

- $p \in \Omega, q \in \Omega, r \in \Omega \implies p \odot (q \oplus r) = (p \odot q) + (p \odot r)$ [Left Distributivity Property]
- $p \in \Omega, q \in \Omega, r \in \Omega \implies (q \oplus r) \odot p = (q \odot p) + (r \odot p)$ [Right Distributivity Property]

Vector Space A vector space is a quadruple $\{\Omega, \oplus, \mathbb{F}, \odot\}$ over a field $\{\mathbb{F}, +, \times\}$ such that $\{\Omega, \oplus\}$ is an abelian group (vector addition), i.e. satisfies

- $p \in \Omega, q \in \Omega \implies p \oplus q \in \Omega$ [Closure Property]
- $p \in \Omega, q \in \Omega, r \in \Omega \implies (p \oplus q) \oplus r = p \oplus (q \oplus r)$ [Associativity Property]
- $\exists 0 \in \Omega, \forall p \in \Omega, p \oplus 0 = 0 \oplus p = p$ [Existence of Identity]
- $\forall p \in \Omega, \exists -p \in \Omega, p \oplus -p = -p \oplus p = 0$ [Existence of Inverses]
- $p \in \Omega, q \in \Omega, \implies p \oplus q = q \oplus p$ [Abelian/Commutativity Property]

and the scalar multiplication $\odot$, vector addition $\oplus$, field addition $+$, and field multiplication $\times$ satisfy

- $\forall \alpha \in \mathbb{F}, p \in \Omega, \alpha \odot p \in \Omega$ [Scalar multiplication Closure]
- $\forall p \in \Omega, 1 \odot p = p$ [Scalar multiplication with Field Identity]
- $\forall \alpha, \beta \in \mathbb{F}, p \in \Omega, \alpha \odot (\beta \odot p) = (\alpha \times \beta) \odot p$ [Scalar, field multiplication Compatibility]
- $\forall \alpha \in \mathbb{F}, p, q \in \Omega, \alpha \odot (p \oplus q) = (\alpha \odot p) \oplus (\alpha \odot q)$ [Scalar multiplication over Vector Addition Distributivity]
- $\forall \alpha, \beta \in \mathbb{F}, p \in \Omega, (\alpha + \beta) \odot p = (\alpha \odot p) \oplus (\beta \odot p)$ [Scalar multiplication over Field Addition Distributivity]

Problem 6. *State the fundamental theorem of algebra.*

Solution:

Every non constant single variable polynomial with complex number coefficients has at least one root. (Implying an n order complex number based polynomial has n complex number roots).

Problem 7. *In a complex field what is the multiplicative inverse of* $2+3i$*, and in quarternion field, what is the product of* $(2+3i)(1+i-4j+5k)$*?*

Solution:

- Complex inverse of $2+3i$

$$\frac{1}{2+3i} = \frac{(2-i3)}{(2+3i)(2-3i)} = \frac{2-3i}{13} = \boxed{\frac{2}{13} - \frac{3}{13}i}$$

-

$$\begin{aligned}(2+3i)(1+i-4j+5k) &= 2+2i-8j+10k+3i+3i^2-12i\cdot j+15i\cdot k \\ &= 2+2i-8j+10k+3i-3-12k-15j = \boxed{-1+5i-23j-2k}\end{aligned}$$

Problem 8. *Prove that the set $\mathbb{Z}_3 = \{0, 1, 2\}$ with modulo 3 arithmetic is a ring. Modulo arithmetic is similar to* clock *arithmetic which is modulo 12 where 12 is same as 0, and therefore $10 + 3$ which would be 13 is same as $13 - 12 = 1$.*

Solution:

Proof. The addition and multiplication tables for $\mathbb{Z}_3$ modulo 3 are

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Ring: A ring is a triple $\{\Omega, \oplus, \odot\}$ such that $\{\Omega, \oplus\}$ is an abelian (also called commutative) group, i.e. satisfies

- $p \in \Omega, q \in \Omega \implies p \oplus q \in \Omega$ [Closure Property]
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The closure, associativity, commutativity, distributivity properties can be all checked from the tables. For the inverses, we can see that 1 and 2 are the additive inverses of each other.

□

Problem 9. *Prove that the set polynomials with coefficients in $\mathbb{R}$ is a vector space over the real field $\mathbb{R}$.*

Solution:

Proof. A vector space is a quadruple $\{\Omega, \oplus, \mathbb{F}, \odot\}$ over a field $\{\mathbb{F}, +, \times\}$ such that $\{\Omega, \oplus\}$ is an abelian group (vector addition), i.e. satisfies

- $p \in \Omega, q \in \Omega \implies p \oplus q \in \Omega$ [Closure Property]
- $p \in \Omega, q \in \Omega, r \in \Omega \implies (p \oplus q) \oplus r = p \oplus (q \oplus r)$ [Associativity Property]
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Using polynomials of type $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, $b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$, and field value α, β, etc. we can easily confirm all the properties of vector addition and scalar multiplication. For instance, the additive identity is the zero polynomial, the negative inverse is the negative of the given polynomial etc. $\square$

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Problem 10. *Draw a unit circle for* $\mathbb{R}^2$ *using* ℓ_2 *norm,* ℓ_1 *norm, and* ℓ_∞ *norm, on the same plot.*

Solution: