Student Name: ____________________

HW#2 CVL851: Special Topics in Transportation, Spring 2023

Problem 1 (10 Points) Give the definitions for Banach spaces and Hilbert spaces, and one example of each.

Problem 2 (20 Points)

1. Find the angle between the vectors $x_1 = [1\ 3\ \ -4\ 0]'$, and $x_1 = [2\ 1\ 4\ \ -1]'$ using the ℓ_2 Hilbert space on $\mathbb{R}^4$.
2. Find the angle between the functions $f_1(x) = x$, and $f_2(x) = x^2$ using the L_2 Hilbert space defined on $x \in [0, 1]$. Are they orthogonal?
3. Find the angle between the functions $f_1(\theta) = \sin\theta$, and $f_2(x) = \cos\theta$ using the L_2 Hilbert space defined on $\theta \in [0, 2\pi]$. Are they orthogonal?
4. Find the norm $\|(x_n)\|$ of the sequence $x_n = 1/\sqrt{2^n}$, $n = 1, 2, \cdots$ in ℓ_2. Notice that this is an infinite sequence, ad hence, just use infinite sum for the norm extending the idea of ℓ_2 from $\mathbb{R}^n$.

Problem 3 (10 Points)

1. Find the maximum value of the function $f(x) = x$ on $(0, 1)$.
2. Find the maximum value of the function

$$f(x) = \begin{cases} x, & \text{if } x \in [0, 1) \\ 0.5, & \text{if } x = 1 \end{cases}$$

   on $[0, 1]$.
3. Find the maximum value of the function $f(x) = x$ on $\mathbb{R}$.
4. Find the maximum value of the function $f(x) = x$ on $[0, 1]$.

Problem 4 (15 Points) Consider the set $X = \{a, b, c, d, e\}$, and the following collection of subsets. Which ones are topologies and which ones are not. Show why.

1. $\mathcal{T}_1 = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$
2. $\mathcal{T}_2 = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$
3. $\mathcal{T}_3 = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{a, b, d, e\}, X\}$

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Problem 5 (5 Points) The set $X_1 = \{a, b, c\}$ has the topology $\mathcal{T}_1 = \{\phi, \{a\}, \{a, b\}, X_1\}$. Similarly, the set $X_2 = \{0, 1\}$ has the topology $\mathcal{T}_2 = \{\phi, \{0\}, \{1\}, X_2\}$. Is the function $f : X_1 \to X_2$, where $f(a) = f(b) = 0$ and $f(c) = 1$ continuous? Show why or why not.

Problem 6 (15 Points) The set $X = \{a, b, c\}$ has the topology $\mathcal{T} = \{\phi, \{a\}, \{a, b\}, X_1\}$. What is the limit of the following sequences? Give reason why for each.

1. $x = (a, a, b, b, b, b, b, b, \cdots, b, b, b, \cdots)$, that is, all bs after first two as.
2. $x = (b, b, a, a, a, a, a, a, \cdots, a, a, a, \cdots)$, that is, all as after first two bs. $x = (a, b, a, b, a, b, \cdots)$, that is, alternate as and bs.

Problem 7 (5 Points) What is a compact set in a metric space, and state the Weirstrass theorem for optimization.
