

Student Name: \_\_\_\_\_

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**HW#2 CVL851: Special Topics in Transportation, Spring 2023**

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**Problem 1** (10 Points) Give the definitions for Banach spaces and Hilbert spaces, and one example of each.

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**Problem 2** (20 Points)

1. Find the angle between the vectors  $x_1 = [1 \ 3 \ -4 \ 0]'$ , and  $x_2 = [2 \ 1 \ 4 \ -1]'$  using the  $\ell_2$  Hilbert space on  $\mathbb{R}^4$ .
2. Find the angle between the functions  $f_1(x) = x$ , and  $f_2(x) = x^2$  using the  $L_2$  Hilbert space defined on  $x \in [0, 1]$ . Are they orthogonal?
3. Find the angle between the functions  $f_1(\theta) = \sin \theta$ , and  $f_2(\theta) = \cos \theta$  using the  $L_2$  Hilbert space defined on  $\theta \in [0, 2\pi]$ . Are they orthogonal?
4. Find the norm  $\|(x_n)\|$  of the sequence  $x_n = 1/\sqrt{2^n}$ ,  $n = 1, 2, \dots$  in  $\ell_2$ . Notice that this is an infinite sequence, and hence, just use infinite sum for the norm extending the idea of  $\ell_2$  from  $\mathbb{R}^n$ .

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**Problem 3** (10 Points)

1. Find the maximum value of the function  $f(x) = x$  on  $(0, 1)$ .
2. Find the maximum value of the function

$$f(x) = \begin{cases} x, & \text{if } x \in [0, 1) \\ 0.5, & \text{if } x = 1 \end{cases}$$

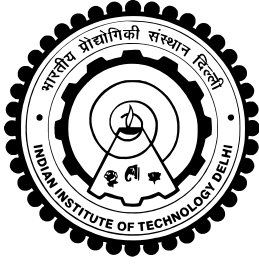
on  $[0, 1]$ .

3. Find the maximum value of the function  $f(x) = x$  on  $\mathbb{R}$ .
4. Find the maximum value of the function  $f(x) = x$  on  $[0, 1]$ .

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**Problem 4** (15 Points) Consider the set  $X = \{a, b, c, d, e\}$ , and the following collection of subsets. Which ones are topologies and which ones are not. Show why.

1.  $\mathcal{T}_1 = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$
  2.  $\mathcal{T}_2 = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$
  3.  $\mathcal{T}_3 = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{a, b, d, e\}, X\}$
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**Problem 5** (5 Points) The set  $X_1 = \{a, b, c\}$  has the topology  $\mathcal{T}_1 = \{\phi, \{a\}, \{a, b\}, X_1\}$ . Similarly, the set  $X_2 = \{0, 1\}$  has the topology  $\mathcal{T}_2 = \{\phi, \{0\}, \{1\}, X_2\}$ . Is the function  $f : X_1 \rightarrow X_2$ , where  $f(a) = f(b) = 0$  and  $f(c) = 1$  continuous? Show why or why not.

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**Problem 6** (15 Points) The set  $X = \{a, b, c\}$  has the topology  $\mathcal{T} = \{\phi, \{a\}, \{a, b\}, X_1\}$ . What is the limit of the following sequences? Give reason why for each.

1.  $x = (a, a, b, b, b, b, b, \dots, b, b, b, \dots)$ , that is, all  $bs$  after first two  $as$ .
  2.  $x = (b, b, a, a, a, a, a, \dots, a, a, a, \dots)$ , that is, all  $as$  after first two  $bs$ .  $x = (a, b, a, b, a, b, \dots)$ , that is, alternate  $as$  and  $bs$ .
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**Problem 7** (5 Points) What is a compact set in a metric space, and state the Weirstrass theorem for optimization.

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