

CVL 851: Assignment #2
SPECIAL TOPICS IN TRANSPORTATION

Due on Monday, February 3rd, 2023

Dr. Pushkin Kachroo

Pushkin Kachroo

February 4, 2023

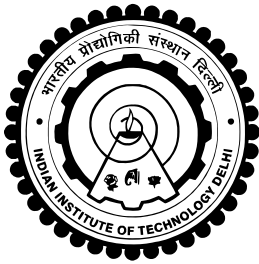


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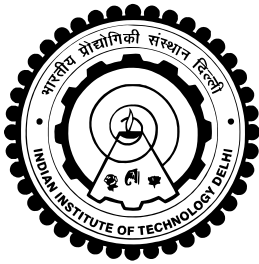
Problem 1. *Give the definitions for Banach spaces and Hilbert spaces, and one example of each.*

Solution:

Banach Space A normed vector space which is a complete metric space in the metric induced by the norm is called a Banach space.

Hilbert Space An inner product space which is a complete metric space in the metric induced by the norm, which in turn is induced by the inner product, is called a Hilbert space.





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- Problem 2.** 1. Find the angle between the vectors $x_1 = [1 \ 3 \ -4 \ 0]'$, and $x_2 = [2 \ 1 \ 4 \ -1]'$ using the ℓ_2 Hilbert space on \mathbb{R}^4 .
2. Find the angle between the functions $f_1(x) = x$, and $f_2(x) = x^2$ using the L_2 Hilbert space defined on $x \in [0, 1]$. Are they orthogonal?
3. Find the angle between the functions $f_1(\theta) = \sin \theta$, and $f_2(\theta) = \cos \theta$ using the L_2 Hilbert space defined on $\theta \in [0, 2\pi]$. Are they orthogonal?
4. Find the norm $\|(x_n)\|$ of the sequence $x_n = 1/\sqrt{2^n}$, $n = 1, 2, \dots$ in ℓ_2 . Notice that this is an infinite sequence, and hence, just use infinite sum for the norm extending the idea of ℓ_2 from \mathbb{R}^n .

Solution:

1.

$$\cos \alpha = \frac{\langle x_1, x_2 \rangle}{\|x_1\| \|x_2\|} = \frac{2 + 3 - 16}{\sqrt{1 + 9 + 16} \sqrt{4 + 1 + 16 + 1}}$$

$$\alpha = 117.38^\circ$$

2.

$$\cos \alpha = \frac{\langle f_1, f_2 \rangle}{\|f_1\| \|f_2\|} = \frac{\int_0^1 f_1(x) f_2(x) dx}{\sqrt{\int_0^1 f_1^2(x) dx} \sqrt{\int_0^1 f_2^2(x) dx}} = \frac{\int_0^1 x^3 dx}{\sqrt{\int_0^1 x^2 dx} \sqrt{\int_0^1 x^4 dx}}$$

$$= \frac{1/4}{\sqrt{1/3} \sqrt{1/5}} = \frac{\sqrt{15}}{4}$$

$$\alpha = 14.478^\circ$$

3.

$$\cos \alpha = \frac{\langle f_1, f_2 \rangle}{\|f_1\| \|f_2\|} = \frac{\int_0^{2\pi} f_1(\theta) f_2(\theta) d\theta}{\sqrt{\int_0^{2\pi} f_1^2(\theta) d\theta} \sqrt{\int_0^{2\pi} f_2^2(\theta) d\theta}} = \frac{\int_0^{2\pi} \sin \theta \cos \theta d\theta}{\sqrt{\int_0^{2\pi} f_1^2(\theta) d\theta} \sqrt{\int_0^{2\pi} f_2^2(\theta) d\theta}}$$

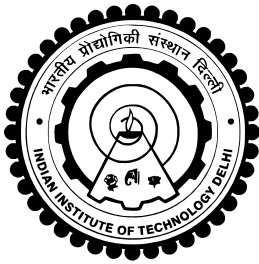
$$= \frac{\int_0^{2\pi} \sin 2\theta d\theta}{2 \sqrt{\int_0^{2\pi} f_1^2(\theta) d\theta} \sqrt{\int_0^{2\pi} f_2^2(\theta) d\theta}} = 0$$

$$\alpha = 90^\circ, \text{ perpendicular}$$

4.

$$\|(x_n)\|_2 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots = 1$$





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- Problem 3.** 1. Find the maximum value of the function $f(x) = x$ on $(0, 1)$.
2. Find the maximum value of the function

$$f(x) = \begin{cases} x, & \text{if } x \in [0, 1) \\ 0.5, & \text{if } x = 1 \end{cases}$$

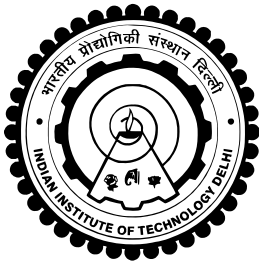
on $[0, 1]$.

3. Find the maximum value of the function $f(x) = x$ on \mathbb{R} .
4. Find the maximum value of the function $f(x) = x$ on $[0, 1]$.

Solution:

1.
2.
3.
4. 1





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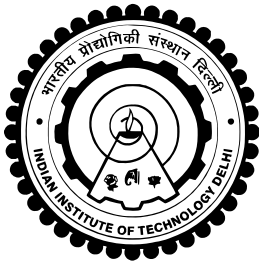
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Problem 4. Consider the set $X = \{a, b, c, d, e\}$, and the following collection of subsets. Which ones are topologies and which ones are not. Show why.

1. $\mathcal{T}_1 = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$
2. $\mathcal{T}_2 = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$
3. $\mathcal{T}_3 = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{a, b, d, e\}, X\}$

Solution:

1. \mathcal{T}_1 is a topology as it satisfies all the axioms of being a topology.
 2. \mathcal{T}_2 is not a topology as $\{a, c, d\} \cup \{b, c, d\} = \{a, b, c, d\} \notin \mathcal{T}_2$
 3. \mathcal{T}_3 is not a topology as $\{a, c, d\} \cap \{a, b, d, e\} = \{a, d\} \notin \mathcal{T}_3$
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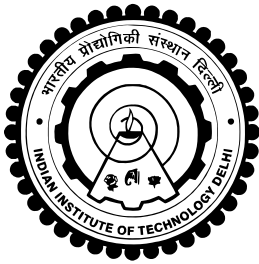
Problem 5. The set $X_1 = \{a, b, c\}$ has the topology $\mathcal{T}_1 = \{\phi, \{a\}, \{a, b\}, X_1\}$. Similarly, the set $X_2 = \{0, 1\}$ has the topology $\mathcal{T}_2 = \{\phi, \{0\}, \{1\}, X_2\}$. Is the function $f : X_1 \rightarrow X_2$, where $f(a) = f(b) = 0$ and $f(c) = 1$ continuous? Show why or why not.

Solution:

A function is continuous if the inverse of every open set in the codomain topology is open in the domain topology.

- $f^{-1}(\phi) = \phi \in \mathcal{T}_1$
- $f^{-1}(X_2) = X_1 \in \mathcal{T}_1$
- $f^{-1}(\{0\}) = \{a, b\} \in \mathcal{T}_1$
- $f^{-1}(\{1\}) = \{c\} \notin \mathcal{T}_1$

Because of $f^{-1}(\{1\}) = \{c\} \notin \mathcal{T}_1$, f is not continuous.



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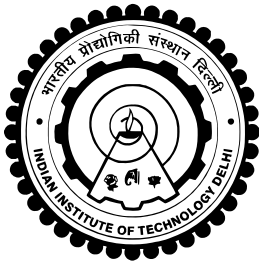
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Problem 6. The set $X = \{a, b, c\}$ has the topology $\mathcal{T} = \{\phi, \{a\}, \{a, b\}, X\}$. What is the limit of the following sequences? Give reason why for each.

1. $x = (a, a, b, b, b, b, b, b, \dots, b, b, b, \dots)$, that is, all bs after first two as.
2. $x = (b, b, a, a, a, a, a, a, \dots, a, a, a, \dots)$, that is, all as after first two bs.
3. $x = (a, b, a, b, a, b, \dots)$, that is, alternate as and bs.

Solution:

1. $\lim_{n \rightarrow \infty} x_n = b$, as any subset that contains b i.e. X and $\{a, b\}$ contain all but finitely many members of the sequence. In fact both contain all the members.
 $\lim_{n \rightarrow \infty} x_n = c$, as the only neighborhood of c is X which contains all the members of the sequence.
 2. $\lim_{n \rightarrow \infty} x_n = a$, as any subset that contains a i.e. X , $\{a\}$, and $\{a, b\}$ contain all but finitely many members of the sequence.
 $\lim_{n \rightarrow \infty} x_n = c$, as the only neighborhood of c is X which contains all the members of the sequence.
 3. $\lim_{n \rightarrow \infty} x_n = b$, as any subset that contains b i.e. X and $\{a, b\}$ contain all but finitely many members of the sequence. In fact both contain all the members.
 $\lim_{n \rightarrow \infty} x_n = c$, as the only neighborhood of c is X which contains all the members of the sequence.
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Problem 7. *What is a compact set in a metric space, and state the Weirstrass theorem for optimization.*

Solution:

Compact Set A set is compact if every sequence in it has a convergent subsequence.

Weirstrass theorem A continuous function on a compact set to R achieves its maximum as well as its minimum.

