CVL 851: Assignment #2

Special Topics in Transportation

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Dr. Pushkin Kachroo

Pushkin Kachroo

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Contents

Problem 1. *Give the definitions for Banach spaces and Hilbert spaces, and one example of each.*

Solution:

Banach Space A normed vector space which is a complete metric space in the metric induced by the norm is called a Banach space.

Hilbert Space An inner product space which is a complete metric space in the metric induced by the norm, which in turn is induced by the inner product, is called a Hilbert space.

Problem 2. 1. *Find the angle between the vectors* $x_1 = [1\ 3\ -4\ 0]'$, *and* $x_1 = [2\ 1\ 4\ -1]'$ *using the* ℓ_2 *Hilbert space on* $\mathbb{R}^4$.

2. *Find the angle between the functions* $f_1(x) = x$, *and* $f_2(x) = x^2$ *using the* L_2 *Hilbert space defined on* $x \in [0, 1]$. *Are they orthogonal?*

3. *Find the angle between the functions* $f_1(\theta) = \sin\theta$, *and* $f_2(x) = \cos\theta$ *using the* L_2 *Hilbert space defined on* $\theta \in [0, 2\pi]$. *Are they orthogonal?*

4. *Find the norm* $\|(x_n)\|$ *of the sequence* $x_n = 1/\sqrt{2^n}$, $n = 1, 2, \cdots$ *in* ℓ_2. *Notice that this is an infinite sequence, ad hence, just use infinite sum for the norm extending the idea of* ℓ_2 *from* $\mathbb{R}^n$.

Solution:

1.
$$\cos\alpha = \frac{\langle x_1, x_2\rangle}{\|x_1\|\|x_2\|} = \frac{2+3-16}{\sqrt{1+9+16}\sqrt{4+1+16+1}}$$
$$\boxed{\alpha = 117.38^\circ}$$

2.
$$\cos\alpha = \frac{\langle f_1, f_2\rangle}{\|f_1\|\|f_2\|} = \frac{\int_0^1 f_1(x)f_2(x)dx}{\sqrt{\int_0^1 f_1^2(x)dx}\sqrt{\int_0^1 f_2^2(x)dx}} = \frac{\int_0^1 x^3dx}{\sqrt{\int_0^1 x^2dx}\sqrt{\int_0^1 x^4dx}}$$
$$= \frac{1/4}{\sqrt{1/3}\sqrt{1/5}} = \frac{\sqrt{15}}{4}$$
$$\boxed{\alpha = 14.478^\circ}$$

3.
$$\cos\alpha = \frac{\langle f_1, f_2\rangle}{\|f_1\|\|f_2\|} = \frac{\int_0^{2\pi} f_1(\theta)f_2(\theta)d\theta}{\sqrt{\int_0^{2\pi} f_1^2(\theta)d\theta}\sqrt{\int_0^{2\pi} f_2^2(\theta)d\theta}} = \frac{\int_0^{2\pi} \sin\theta\cos\theta d\theta}{\sqrt{\int_0^{2\pi} f_1^2(\theta)d\theta}\sqrt{\int_0^{2\pi} f_2^2(\theta)d\theta}}$$
$$= \frac{\int_0^{2\pi} \sin 2\theta d\theta}{2\sqrt{\int_0^{2\pi} f_1^2(\theta)d\theta}\sqrt{\int_0^{2\pi} f_2^2(\theta)d\theta}} = 0$$
$$\boxed{\alpha = 90^\circ,\ \text{perpendicular}}$$

4.
$$\boxed{\|(x_n)\|_2 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = 1}$$

Problem 3. 1. *Find the maximum value of the function* $f(x) = x$ *on* $(0, 1)$.

2. *Find the maximum value of the function*

$$f(x) = \begin{cases} x, & \text{if } x \in [0,1) \\ 0.5, & \text{if } x = 1 \end{cases}$$

on $[0, 1]$.

3. *Find the maximum value of the function* $f(x) = x$ *on* $\mathbb{R}$.

4. *Find the maximum value of the function* $f(x) = x$ *on* $[0, 1]$.

Solution:

1. Does not exist
2. Does not exist
3. Does not exist
4. 1

Problem 4. *Consider the set* $X = \{a, b, c, d, e\}$*, and the following collection of subsets. Which ones are topologies and which ones are not. Show why.*

1. $\mathcal{T}_1 = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$
2. $\mathcal{T}_2 = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$
3. $\mathcal{T}_3 = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{a, b, d, e\}, X\}$

Solution:

1. $\mathcal{T}_1$ is a topology as it satisfies all the axioms of being a topology.
2. $\mathcal{T}_2$ is not a topology as $\{a, c, d\} \cup \{b, c, d\} = \{a, b, c, d\} \notin \mathcal{T}_2$
3. $\mathcal{T}_3$ is not a topology as $\{a, c, d\} \cap \{a, b, d, e\} = \{a, d\} \notin \mathcal{T}_3$

Problem 5. *The set* $X_1 = \{a, b, c\}$ *has the topology* $\mathcal{T}_1 = \{\phi, \{a\}, \{a, b\}, X_1\}$. *Similarly, the set* $X_2 = \{0, 1\}$ *has the topology* $\mathcal{T}_2 = \{\phi, \{0\}, \{1\}, X_2\}$. *Is the function* $f : X_1 \to X_2$, *where* $f(a) = f(b) = 0$ *and* $f(c) = 1$ *continuous? Show why or why not.*

Solution:

A function is continuous if the inverse of every open set in the codomain topology is open in the domain topology.

- $f^{-1}(\phi) = \phi \in \mathcal{T}_1$
- $f^{-1}(X_2) = X_1 \in \mathcal{T}_1$
- $f^{-1}(\{0\}) = \{a, b\} \in \mathcal{T}_1$
- $f^{-1}(\{1\}) = \{c\} \notin \mathcal{T}_1$

Because of $f^{-1}(\{1\}) = \{c\} \notin \mathcal{T}_1$, f is not continuous.

Problem 6. *The set $X = \{a, b, c\}$ has the topology $\mathcal{T} = \{\phi, \{a\}, \{a, b\}, X\}$. What is the limit of the following sequences? Give reason why for each.*

1. $x = (a, a, b, b, b, b, b, b, \cdots, b, b, b, \cdots)$, *that is, all bs after first two as.*
2. $x = (b, b, a, a, a, a, a, a, \cdots, a, a, a, \cdots)$, *that is, all as after first two bs.*
3. $x = (a, b, a, b, a, b, \cdots)$, *that is, alternate as and bs.*

Solution:

1. $\lim_{n\to\infty} x_n = b$, as any subset that contains b i.e. X and $\{a, b\}$ contain all but finitely many members of the sequence. In fact both contain all the members.
   $\lim_{n\to\infty} x_n = c$, as the only neighborhood of c is X which contains all the members of the sequence.
2. $\lim_{n\to\infty} x_n = a$, as any subset that contains a i.e. X, $\{a\}$, and $\{a, b\}$ contain all but finitely many members of the sequence.
   $\lim_{n\to\infty} x_n = c$, as the only neighborhood of c is X which contains all the members of the sequence.
3. $\lim_{n\to\infty} x_n = b$, as any subset that contains b i.e. X and $\{a, b\}$ contain all but finitely many members of the sequence. In fact both contain all the members.
   $\lim_{n\to\infty} x_n = c$, as the only neighborhood of c is X which contains all the members of the sequence.

Problem 7. *What is a compact set in a metric space, and state the Weirstrass theorem for optimization.*

Solution:

Compact Set A set is compact if every sequence in it has a convergent subsequence.

Weirstrass theorem A constinuous function on a compact set to R achieves its maximum as well as its minimum.