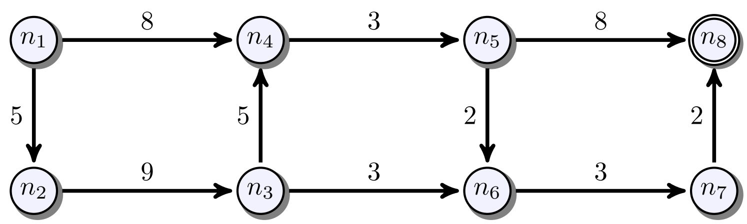
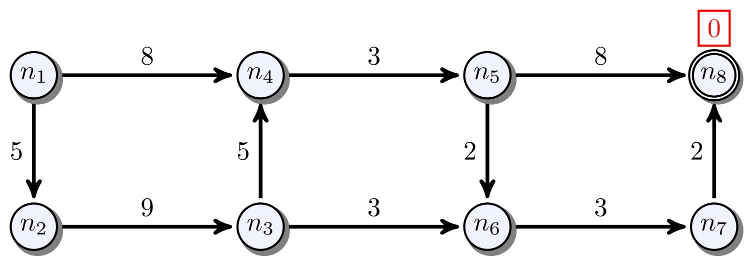
Student Name: ____________________

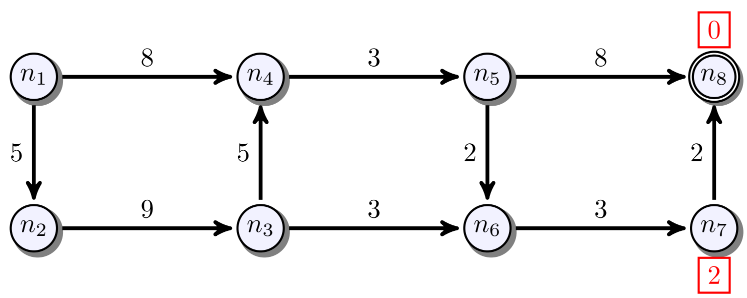
HW#4 CVL851: Special Topics in Transportation, Spring 2023

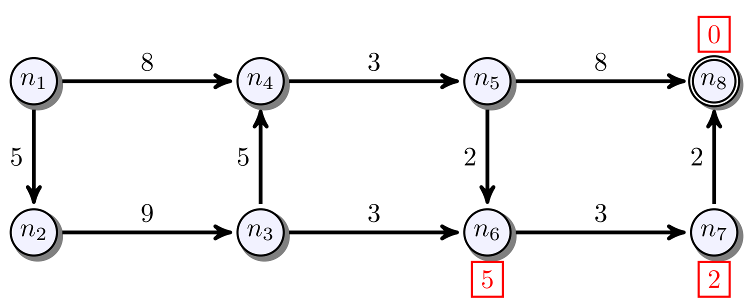
Problem 1 (20 Points) The final state in the figure is n_8. Using Bellman's principle for Dynamic Programming, find the value function for every state, and using that find the shortest path sequentially from n_1 to n_8, and also from n_3 to n_8.

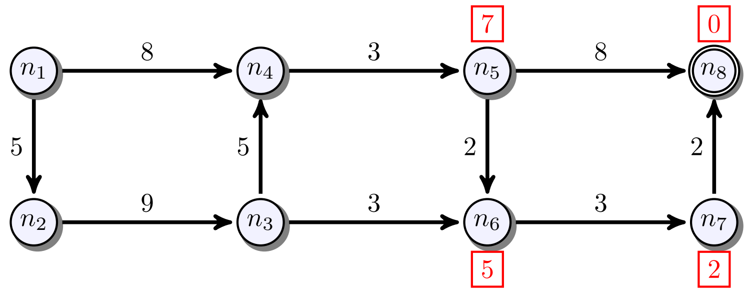


Solution:





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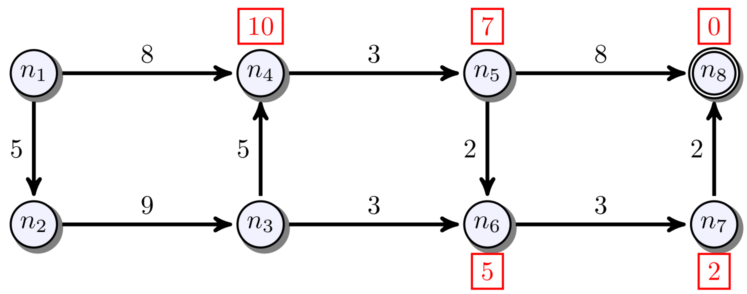
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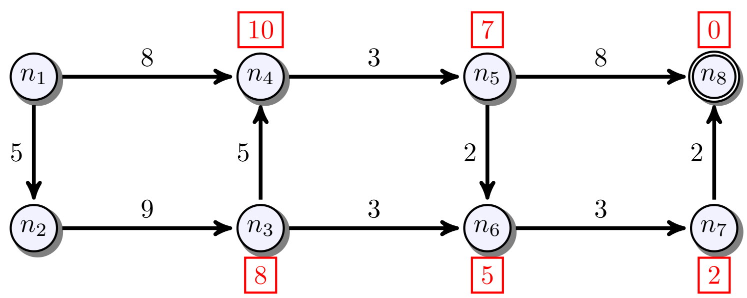
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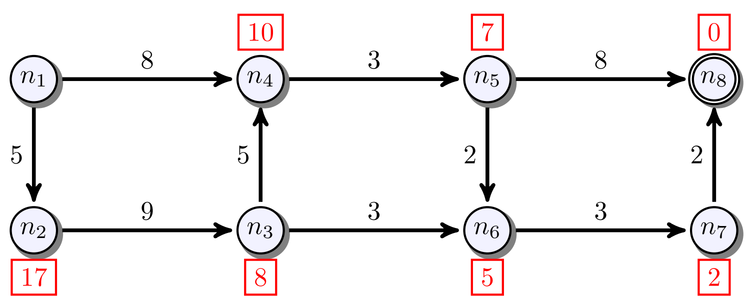
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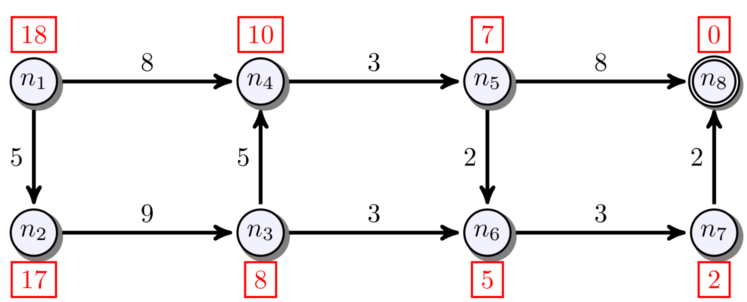
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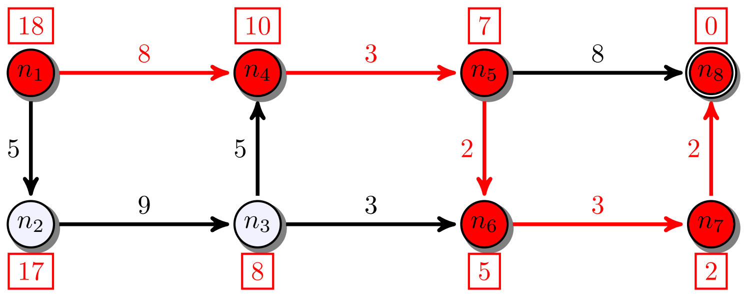

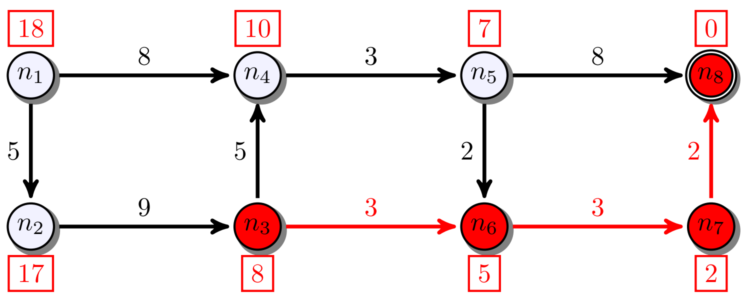

Problem 2 (20 Points)

1. Given the optimization problem

$$\text{minimize} \int_{t_0}^{t_f} f(t, x(t), \dot{x}(t))dt, \quad x(t_0) = x_0, \quad x(t_f) = x_f$$

   derive the Euler Lagrange condition to be solved.

2. Solve the problem

$$\text{minimize} \int_{0}^{\pi/2} (\dot{x}^2(t) - x^2(t))dt, \quad x(0) = 0, \quad x(\pi/2) = 1$$

Solution:

1. The Gateaux derivative must be zero in any direction from $x(\cdot)$ to any $\delta(\cdot)$. Hence, we write the functional as:

$$F(\epsilon) = \int_{t_0}^{t_f} f(t, x(t) + \epsilon\delta(t), \dot{x}(t) + \epsilon\dot{\delta}(t))dt$$

   To make Gateaux derivative to be zero, we take the derivative with respect to ϵ and equate it to zero. We use chain rule on the right hand side.

$$\frac{dF(\epsilon)}{d\epsilon} = \int_{t_0}^{t_f} [f_x\delta(t) + f_{\dot{x}}\dot{\delta}(t)]dt = 0$$

   Using integration by parts, we get

$$\frac{dF(\epsilon)}{d\epsilon} = \int_{t_0}^{t_f} f_x\delta(t)dt + \cancelto{0}{[f_{\dot{x}}\delta(t)]_{t_0}^{t_f}} - \int_{t_0}^{t_f} [\frac{d}{dt}f_{\dot{x}}]\delta(t)dt = 0$$

$$\forall\delta(\cdot), \int_{t_0}^{t_f} [f_x + \frac{d}{dt}f_{\dot{x}}]\delta(t)dt = 0$$

$$\implies \text{Euler Lagrange Equation}$$

$$\boxed{f_x + \frac{d}{dt}f_{\dot{x}} = 0}$$

2. Given the optimization problem

$$f(t, x(t), \dot{x(t)}) = \dot{x}^2(t) - x^2(t)$$

$$f_x = -2x, \quad f_{\dot{x}} = 2\dot{x}$$

$$\frac{d}{dt}f_{\dot{x}} = 2\ddot{x}$$

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Euler Lagrange Equation:

$$2\ddot{x} = -2x$$

$$\frac{d}{dt^2}x(t) = -x(t), \quad x(0) = 0,\ x(\pi/2) = 1$$

$$x(t) = A\sin(t) + B\cos(t)$$

Apply the boundary conditions $x(0) = 0,\ x(\pi/2) = 1$ to obtain

$$\boxed{x(t) = \sin(t)}$$

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HW#4 CVL851: Special Topics in Transportation, Spring 2023

Problem 3 (50 Points)

1. What is the probability of getting three sixes in three rolls of a dice?
2. A subway train made up of n cars is boarded by r passengers ($r \leq n$), each entering a car completely at random. What is the probability of the passengers all ending up in different cars?
3. A batch of 100 manufactured items is checked by an inspector, who examines 10 items selected at random. If none of the 10 items is defective, he accepts the whole batch. Otherwise, the batch is subjected to further inspection. What is the probability that a batch containing 10 defective items will be accepted?
4. What is the probability that two playing cards picked at random from a full deck are both aces?
5. What is the probability that each of four bridge players holds an ace?

Solution:

1.

$$\boxed{P(A) = \frac{1}{6^3} = \frac{1}{216}}$$

2.

$$\boxed{P(A) = \frac{n(n-1)\cdots(n-(r-1))}{n^r}}$$

3. The number of ways to select 10 out of 100 items is

$$N = C_{10}^{100} = \frac{100!}{90!10!}$$

The number of ways to select 10 non-defective items is

$$N_d = C_{10}^{90} = \frac{90!}{80!10!}$$

Hence,

$$\boxed{P(A) = \frac{N_d}{N} = \frac{90!90!}{80!100!} = \frac{90 \cdot 89 \cdots 81}{100 \cdot 99 \cdots 91}}$$

4.

$$\boxed{P(A) = \frac{C_2^4}{C_4^{52}} = \frac{1}{221}}$$

5.

$$\boxed{P(A) = \frac{4!\dfrac{48!}{12!12!12!12!}}{\dfrac{52!}{13!13!13!13!}} = \frac{24(13^4)}{52 \cdot 51 \cdot 50 \cdot 49} \sim 0.105}$$

Problem 4 (30 Points)

1. Define measurable space, measure space, and probability space.

2. Define random variable.

Solution:

1. **Measurable Space** A tuple (Ω, Σ) is called a measurable space, if the set of subsets, Σ of the set Ω is a sigma algebra, i.e. it satisfies:
   (a) $\phi \in \Sigma$
   (b) $A \in \Sigma \implies A^c \in \Sigma$
   (c) For any countable I,
   $\underset{i \in I}{\cup} A_i \in \Sigma$, if $\forall i \in I,\ A_i \in \Sigma$

   Measure Space A triple (Ω, Σ, μ) is called a measure space, if (Ω, Σ) is a measurable space and $\mu : \Sigma \to \mathbb{R}^+$ is a measure, i.e. it satisfies:
   (a) $\mu(\phi) = 0$
   (b) *Countable Additivity*: For any countable mutually disjoint collection $A_i \in \Sigma$,
   $\mu\left(\overset{\infty}{\underset{i=1}{\cup}} A_i\right) = \overset{\infty}{\underset{i=1}{\Sigma}} \mu(A_i)$

   Probability Space A triple (Ω, Σ, P) is called a probability space if it is a normalized measure space, i.e. $P(\Omega) = 1$.

2. Random variable is a measurable function from a probability space to $\mathbb{R}$, the measurable space of real numbers.