CVL 851: Minor Exam #1

Special Topics in Transportation

Due on Monday, February 6^{th}, 2023

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Contents

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Problem 1. *Given the following table for a groupoid* $\{\Omega, \odot\}$.

$\odot$	a	b	c
a	b	c	a
b	c	a	b
c	a	b	c

1. *Find the identity element of the groupoid.*

   c

2. *Fill the following table*

$(a \odot b) \odot c =$	c	$a \odot (b \odot c) =$	c
$(a \odot c) \odot b =$	c	$a \odot (c \odot b) =$	c
$(b \odot c) \odot a =$	c	$b \odot (c \odot a) =$	c
$(b \odot a) \odot c =$	c	$b \odot (a \odot c) =$	c
$(c \odot a) \odot b =$	c	$c \odot (a \odot b) =$	c
$(c \odot b) \odot a =$	c	$c \odot (b \odot a) =$	c

   Is this groupoid a semigroup? *yes, being associative*

3. *Is this groupoid a monoid?* *yes, because has the identity element*

4. *Fill the following table*

$a \odot b =$	c	$b \odot a =$	c
$a \odot c =$	a	$c \odot a =$	a
$b \odot c =$	b	$c \odot b =$	b

   Is this groupoid abelian? *yes, being associative*

5. *Find the inverses of* a, b, *and* c. *Is this groupoid a group?*

   $a^{-1} = b,\ b^{-1} = a,\ c^{-1} = c$

   yes, being a monoid with inverses

Problem 2. 1. *Find the angle between the vectors* $x_1 = [1\ 0\ -1\ 0]'$, *and* $x_1 = [-1\ 1\ 1\ -1]'$ *using the* ℓ_2 *Hilbert space on* $\mathbb{R}^4$.

2. *Find the angle between the functions* $f_1(\theta) = \sin\theta$, *and* $f_2(\theta) = \cos\theta$ *using the* L_2 *Hilbert space defined on* $\theta \in [0, \pi]$. *Are they orthogonal?*

3. *Find the* L_2 *limit of the series of functions* $f_n(x) = x^n$ *as* $n \to \infty$ *defined on* $[0, 1]$.

4. *Find the norm* $\|(x_n)\|$ *of the sequence* $x_n = 1/\sqrt{3^n}$, $n = 1, 2, \cdots$ *in* ℓ_2. *Notice that this is an infinite sequence, and hence, just use infinite sum for the norm extending the idea of* ℓ_2 *from* $\mathbb{R}^n$.

Solution:

1. $$\theta = \arccos\left(\frac{\langle x_1, x_2\rangle}{\|x_1\|\|x_2\|}\right) = \arccos\left(\frac{-2}{\sqrt{2}\sqrt{2}}\right) = \arccos(-1) = \boxed{\pi \text{ radians}} \tag{1}$$

2. $$\begin{aligned}\theta &= \arccos\left(\frac{\langle f_1, f_2\rangle}{\|f_1\|\|f_2\|}\right) = \arccos\left(\frac{\int_0^\pi \sin\theta\cos\theta d\theta}{\|f_1\|\|f_2\|}\right)\\ &= \arccos\left(\frac{\int_0^\pi \sin 2\theta d\theta}{2\|f_1\|\|f_2\|}\right) = \arccos(0) = \boxed{\pi/2 \text{ radians}}\end{aligned} \tag{2}$$

   yes, they are orthogonal

3. $f_n(x) = x^n \to 0$

4. $$\sum x_n = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots = \frac{1/3}{1 - (1/3)} = \boxed{\frac{1}{2}} \tag{3}$$

Problem 3. *The set* $X_1 = \{a, b, c\}$ *has the topology* $\mathcal{T}_1 = \{\phi, \{a\}, \{a, b\}, X_1\}$. *Similarly, the set* $X_2 = \{0, 1\}$ *has the topology* $\mathcal{T}_2 = \{\phi, \{0\}, \{1\}, X_2\}$. *Is the function* $f : X_1 \to X_2$, *where* $f(a) = 0$ *and* $f(b) = f(c) = 1$ *continuous? Show why or why not.*

Solution:

$f^{-1}(1) = \{b, c\} \notin \mathcal{T}_1 \implies f$ is not continuous.

Problem 4. *The set* $X = \{a, b, c\}$ *has the topology* $\mathcal{T} = \{\phi, \{a\}, \{b, c\}, X\}$. *What is the limit of the following sequences? Give reason why for each.*

1. $x = (a, a, b, b, b, b, b, b, \cdots, b, b, b, \cdots)$, *that is, all bs after first two as.*
2. $x = (b, b, a, a, a, a, a, a, \cdots, a, a, a, \cdots)$, *that is, all as after first two bs.*
3. $x = (a, b, a, b, a, b, \cdots)$, *that is, alternate as and bs.*

Solution:

1. $x \to b$ and also $x \to c$. This is because any open set containing c also contains b.
2. $x \to a$
3. This sequence does not converge.