

CVL 851: Minor Exam #1
SPECIAL TOPICS IN TRANSPORTATION

Due on Monday, February 6th, 2023

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February 15, 2023

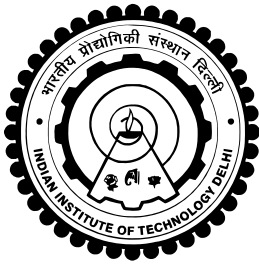


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Minor#1: Solution CVL851: Special Topics in Transportation, Spring 2023

Contents

Problem 1	3
Problem 2	4
Problem 3	5
Problem 4	6



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Problem 1. Given the following table for a groupoid $\{\Omega, \odot\}$.

\odot	a	b	c
a	b	c	a
b	c	a	b
c	a	b	c

1. Find the identity element of the groupoid.

c

2. Fill the following table

$(a \odot b) \odot c =$	c	$a \odot (b \odot c) =$	c
$(a \odot c) \odot b =$	c	$a \odot (c \odot b) =$	c
$(b \odot c) \odot a =$	c	$b \odot (c \odot a) =$	c
$(b \odot a) \odot c =$	c	$b \odot (a \odot c) =$	c
$(c \odot a) \odot b =$	c	$c \odot (a \odot b) =$	c
$(c \odot b) \odot a =$	c	$c \odot (b \odot a) =$	c

Is this groupoid a semigroup? $yes, being\ associative$

3. Is this groupoid a monoid? $yes, because\ has\ the\ identity\ element$

4. Fill the following table

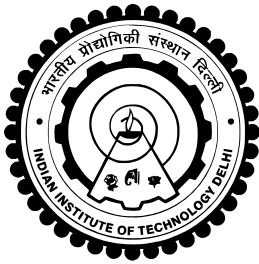
$a \odot b =$	c	$b \odot a =$	c
$a \odot c =$	a	$c \odot a =$	a
$b \odot c =$	b	$c \odot b =$	b

Is this groupoid abelian? $yes, being\ associative$

5. Find the inverses of a , b , and c . Is this groupoid a group?

$a^{-1} = b, b^{-1} = a, c^{-1} = c$

$yes, being\ a\ monoid\ with\ inverses$



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- Problem 2.** 1. Find the angle between the vectors $x_1 = [1 \ 0 \ -1 \ 0]'$, and $x_2 = [-1 \ 1 \ 1 \ -1]'$ using the ℓ_2 Hilbert space on \mathbb{R}^4 .
2. Find the angle between the functions $f_1(\theta) = \sin \theta$, and $f_2(\theta) = \cos \theta$ using the L_2 Hilbert space defined on $\theta \in [0, \pi]$. Are they orthogonal?
3. Find the L_2 limit of the series of functions $f_n(x) = x^n$ as $n \rightarrow \infty$ defined on $[0, 1]$.
4. Find the norm $\|(x_n)\|$ of the sequence $x_n = 1/\sqrt{3^n}$, $n = 1, 2, \dots$ in ℓ_2 . Notice that this is an infinite sequence, and hence, just use infinite sum for the norm extending the idea of ℓ_2 from \mathbb{R}^n .

Solution:

1.

$$\theta = \arccos\left(\frac{\langle x_1, x_2 \rangle}{\|x_1\| \|x_2\|}\right) = \arccos\left(\frac{-2}{\sqrt{2}\sqrt{2}}\right) = \arccos(-1) = \boxed{\pi \text{ radians}} \tag{1}$$

2.

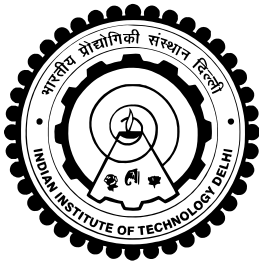
$$\begin{aligned} \theta &= \arccos\left(\frac{\langle f_1, f_2 \rangle}{\|f_1\| \|f_2\|}\right) = \arccos\left(\frac{\int_0^\pi \sin \theta \cos \theta d\theta}{\|f_1\| \|f_2\|}\right) \\ &= \arccos\left(\frac{\int_0^\pi \sin 2\theta d\theta}{2\|f_1\| \|f_2\|}\right) = \arccos(0) = \boxed{\pi/2 \text{ radians}} \end{aligned} \tag{2}$$

yes, they are orthogonal

3. $f_n(x) = x^n \rightarrow 0$

4.

$$\sum x_n = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots = \frac{1/3}{1 - (1/3)} = \boxed{\frac{1}{2}} \tag{3}$$



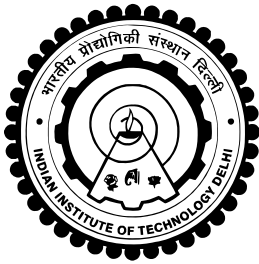
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Minor#1: Solution CVL851: Special Topics in Transportation, Spring 2023

Problem 3. The set $X_1 = \{a, b, c\}$ has the topology $\mathcal{T}_1 = \{\phi, \{a\}, \{a, b\}, X_1\}$. Similarly, the set $X_2 = \{0, 1\}$ has the topology $\mathcal{T}_2 = \{\phi, \{0\}, \{1\}, X_2\}$. Is the function $f : X_1 \rightarrow X_2$, where $f(a) = 0$ and $f(b) = f(c) = 1$ continuous? Show why or why not.

Solution:

$$\underline{f^{-1}(1) = \{b, c\} \notin \mathcal{T}_1} \implies f \text{ is not continuous.}$$



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Problem 4. The set $X = \{a, b, c\}$ has the topology $\mathcal{T} = \{\emptyset, \{a\}, \{b, c\}, X\}$. What is the limit of the following sequences? Give reason why for each.

1. $x = (a, a, b, b, b, b, b, \dots, b, b, b, \dots)$, that is, all bs after first two as.
2. $x = (b, b, a, a, a, a, a, \dots, a, a, a, \dots)$, that is, all as after first two bs.
3. $x = (a, b, a, b, a, b, \dots)$, that is, alternate as and bs.

Solution:

1. $x \rightarrow b$ and also $x \rightarrow c$. This is because any open set containing c also contains b .
2. $x \rightarrow a$
3. This sequence does not converge.

