ECE 5404 Sp'04 HW#1 Solution

(2.1) Leibniz theorem: $\frac{d}{dt}\int_{h_2(t)}^{h_1(t)} f(t,s)ds = f(t, h_1(t))\frac{d}{dt}h_1(t) - f(t, h_2(t))\frac{d}{dt}h_2(t) + \int_{h_2(t)}^{h_1(t)} \frac{\partial}{\partial t} f(t,s)ds$

verification:

Now $x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau$

$\therefore x(t_0) = \Phi(t_0,t_0)x_0 + \cancelto{0}{\int_{t_0}^{t_0} \Phi(t,\tau)B(\tau)u(\tau)d\tau} = I x_0 = x_0$

Initial condition verified.

$$
\begin{aligned}
\dot{x}(t) &= \dot{\Phi}(t,t_0)x_0 + \frac{d}{dt}\int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau \\
&= A(t)\Phi(t,t_0)x_0 + \Phi(t,t)B(t)u(t) + \int_{t_0}^{t} \frac{\partial}{\partial t}\Phi(t,\tau)B(\tau)u(\tau)d\tau \\
&= A(t)\Phi(t,t_0)x_0 + B(t)u(t) + \int_{t_0}^{t} A(t)\Phi(t,\tau)B(\tau)u(\tau)d\tau \\
&= A(t)\left[\Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau\right] + B(t)u(t) \\
&= A(t)x(t) + B(t)u(t)
\end{aligned}
$$

$\therefore$ Dynamics verified

(2.2) $e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \ldots$

(a) $e^{A0} = I + 0 + 0 + \ldots = I$

(b) $e^{-At} = I - At + \frac{A^2t^2}{2!} - \frac{A^3t^3}{3!} + \ldots$

$\therefore \frac{d}{dt}e^{-At} = 0 - A + \frac{1}{2!}2A^2t - \frac{3A^3t^2}{3!} = -A\left[I - A^2t + \frac{1}{2!}A^2t^2 - \frac{1}{3!}A^3t^3 + \ldots\right] = -e^{-At}A$

(c) $e^{At}e^{-A\tau} = \left[I + At + \frac{1}{2!}A^2t^2 + \ldots\right]\left[I - At + \frac{1}{2!}A^2t^2 - \ldots\right]$

$= I + A(t-\tau) + \frac{1}{2!}A^2(t-\tau)^2 + \frac{1}{3!}A^3(t-\tau)^3 + \ldots$ (by collecting terms)

$= e^{A(t-\tau)}$

(2.3) $\dot{x} = Ax + Bu$

$\dot{x}(t) - Ax(t) = Bu(t)$

$e^{-At}\dot{x}(t) - e^{-At}Ax(t) = e^{-At}Bu(t)$

$\frac{d}{dt}\left[e^{-At}x(t)\right] = e^{-At}Bu(t)$

Integrating both sides

$$e^{-At}x(t) - x(0) = \int_0^t e^{-A\tau} B u(\tau)\, d\tau$$

$$\text{or} \quad x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$$

2.4

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

(a) $$y(t) = \int_0^t C e^{A(t-\tau)} B u(\tau)\, d\tau + Du(t) + Ce^{At}x(0)$$

Impulse response

$$g(t) = \left[\int_{0^-}^t C e^{A(t-\tau)} B \delta(\tau)\, d\tau + D\delta(t)\right] 1(t)$$

$$g(t) = \left[C e^{At} B + D\delta(t)\right] 1(t)$$

(b) $$G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\left[Ce^{At}B + D\delta(t)\right]1(t)$$
$$= C\,\mathcal{L}[e^{At}]B + D\,\mathcal{L}\{\delta(t)\}$$
$$= C(sI-A)^{-1}B + D$$

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(a) $$G(s) = C(sI-A)^{-1}B = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} s & -1 \\ 4 & s+5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{(s+1)(s+4)} \begin{bmatrix} 1 \\ 10s \end{bmatrix}$$

(b) $p_1 = -1$, $p_2 = -4$; modes $m_1 = e^{-t}$, $m_2 = e^{-4t}$

For zeroes rank $[G(s)] < 1$

$\therefore$ no zeroes

(c) $$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{ \begin{matrix} \frac{1}{(s+1)(s+4)} \\ \frac{10s}{(s+1)(s+4)} \end{matrix} \right\} = \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{3(s+1)} - \frac{1}{3(s+4)} \\ \frac{-10}{3(s+1)} + \frac{40}{3(s+4)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{-t}}{3} - \frac{e^{-4t}}{3} \\ -\frac{10}{3}e^{-t} + \frac{40}{3}e^{-4t} \end{bmatrix}$$