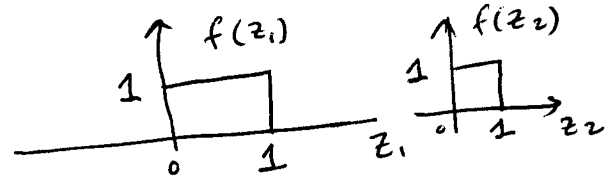
ECE5404 HW#3 SOLUTION




3.1 (a) $E(z_1) = \int_{-\infty}^{+\infty} z_1 f(z_1)\,dz_1 = \int_0^1 z_1\,dz_1 = \left[\frac{z_1^2}{2}\right]_0^1 = \frac{1}{2}$

$= E(z_2)$

$$E[x] = \begin{bmatrix} E[z_1] + E[z_2] \\ E[z_1] - E[z_2] \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(b) $\Sigma_x = E\{[x - m_x][x - m_x]^T\} = E[xx^T] - m_x m_x^T$

$$m_x m_x^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$E[xx^T] = E\left[\begin{bmatrix} z_1 + z_2 \\ z_1 - z_2 \end{bmatrix}\begin{bmatrix} z_1 + z_2 & z_1 - z_2 \end{bmatrix}\right] = E\begin{bmatrix} z_1^2 + z_2^2 + 2z_1z_2 & z_1^2 - z_2^2 \\ z_1^2 - z_2^2 & z_1^2 + z_2^2 - 2z_1z_2 \end{bmatrix}$$

$$E[z_1^2] = \int_{-\infty}^{+\infty} z_1^2 f(z_1)\,dz_1 = \int_0^1 z_1^2\,dz_1 = \left[\frac{z_1^3}{3}\right]_0^1 = \frac{1}{3} = E[z_2^2]$$

$E[z_1z_2] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} z_1 z_2 f(z_1, z_2)\,dz_1\,dz_2$ ; $\because$ $z_1$ and $z_2$ are uncorrelated,

we have $E[z_1z_2] = E[z_1]E[z_2] = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$

$$E[xx^T] = \begin{bmatrix} E[z_1^2] + E[z_2^2] + 2E[z_1z_2] & E[z_1^2] - E[z_2^2] \\ E[z_1^2] - E[z_2^2] & E[z_1^2] + E[z_2^2] - 2E[z_1z_2] \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} + \frac{1}{3} + \frac{1}{2} & 0 \\ 0 & \frac{1}{3} + \frac{1}{3} - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 7/6 & 0 \\ 0 & \frac{1}{6} \end{bmatrix}$$

$$\Sigma_x = \begin{bmatrix} 7/6 & 0 \\ 0 & 1/6 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/6 & 0 \\ 0 & 1/6 \end{bmatrix}$$

(c) $\sigma_{x_1}^2 = E[(x_1 - m_{x_1})^2] = E[x_1^2] - m_{x_1}^2 = 7/6 - 1 = 1/6$

(3.2) $E\left[\{x(t)-m_x(t)\}\{x(t)-m_x(t)\}^T\right] = E\left[\{x(t)-m_x(t)\}\{x^T(t)-m_x^T(t)\}\right]$

$= E\left[x(t)x^T(t) - m_x(t)x^T(t) - x(t)m_x^T(t) + m_x(t)m_x^T(t)\right]$

$= E[x(t)x^T(t)] - m_x(t)E[x^T(t)] - E[x(t)]m_x^T(t) + m_x(t)m_x^T(t)$

$= \Sigma_x(t) - m_x(t)m_x^T(t) - \cancel{m_x(t)m_x^T(t)} + \cancel{m_x(t)m_x^T(t)}$

$= \Sigma_x(t) - m_x(t)m_x^T(t)$

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(3.3) (a) $R_w(\tau) = \sigma_w^2 e^{-a|\tau|}$

$$S_w(\omega) = \int_{-\infty}^{+\infty} \sigma_w^2 e^{-a|\tau|} e^{-j\omega\tau}\,d\tau = \int_{-\infty}^{0} \sigma_w^2 e^{(a-j\omega)\tau}\,d\tau + \int_{0}^{+\infty} \sigma_w^2 e^{-(a+j\omega)\tau}\,d\tau$$

$$= \sigma_w^2\left[\left[\frac{1\cdot e^{(a-j\omega)\tau}}{(a-j\omega)}\right]_{-\infty}^{0} + \left[-\frac{1\cdot e^{-(a+j\omega)\tau}}{(a+j\omega)}\right]_{0}^{\infty}\right]$$

$$= \sigma_w^2\left[\frac{1}{(a-j\omega)} + \frac{1}{(a+j\omega)}\right] = \frac{2a\sigma_w^2}{a^2+\omega^2}$$

(b) $\because w(t)$ has zero mean, we have $C_w(\tau) = R_w(\tau)$

$$\therefore \tau_c = \frac{\int_0^\infty C_w(\tau)\,d\tau}{C_w(0)} = \frac{\int_0^\infty \sigma_w^2 e^{-a|\tau|}\,d\tau}{\sigma_w^2} = \int_0^\infty e^{-a\tau}\,d\tau = \left[-\frac{e^{-a\tau}}{a}\right]_0^\infty$$

$$= \frac{1}{a}$$

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(3.4) $\dot{x}(t) = -2x(t) + 3w(t)$ ; $y(t) = 4x(t)$ ; $S_w(\omega) = 16$ , $x(0) = 0$

(a) $\Sigma_x(t) = e^{At}\Sigma_x(0)e^{A^Tt} + \int_0^t e^{A\gamma} B S_w B^T e^{A^T\gamma}\,d\gamma$

Here $A = -2$ ; $B = 3$ , $S_w = 16$

also $\Sigma_x(0) = E[x^2(0)] = 0$

$$\therefore \Sigma_x(t) = \int_0^t e^{-2\gamma}\cdot 3\cdot 16\cdot 3\, e^{-2\gamma}\,d\gamma = \int_0^t 144\, e^{-4\gamma}\,d\gamma = -36\left[e^{-4\gamma}\right]_0^t$$

$\therefore \Sigma_x(t) = 36\left(1 - e^{-4t}\right)$

$$\Sigma_x(t) = \mathcal{E}_x(t) - \cancel{m_x(t)} m_x^T(t)$$

$$= \underline{36(1-e^{-4t})}$$

(b) $$\dot{\mathcal{E}}_x(t) = A\mathcal{E}_x(t) + \mathcal{E}_x(t)A^T + BS_w B^T$$

$$= -2\mathcal{E}_x(t) - 2\mathcal{E}_x(t) + 144$$

$$\dot{\mathcal{E}}_x(t) = -4\mathcal{E}_x(t) + 144 \quad ; \quad \mathcal{E}_x(0) = 0$$

$$\therefore \mathcal{E}_x(t) = \int_0^t e^{-4(t-z)} \cdot 144 \cdot dz = 144\int_t^0 -e^{-4y}\,dy = 144\int_0^t e^{-4y}\,dy$$

$$= -36\left[e^{-4y}\right]_0^t = \underline{36(1-e^{-4t})} = \Sigma_x(t)$$

(c) $$\mathcal{E}_y(t) = C\mathcal{E}_x(t)C^T = 16 \cdot 36(1-e^{-4t}) = \underline{576(1-e^{-4t})} = \Sigma_y(t)$$

(d) $$A\mathcal{E}_x(\infty) + \mathcal{E}_x(\infty)A^T + BS_x B^T = 0$$

$$-2\mathcal{E}_x(\infty) - 2\mathcal{E}_x(\infty) + 144 = 0$$

$$\mathcal{E}_x(\infty) = \frac{144}{4} = \underline{36}$$