ECE 5404 HW #6 Solutions.

(6.5) $\dot{x}(t) = Ax(t) + B_u u(t) = 2x(t) + u(t)$; $A = 2$, $B_u = 1$, $t_f = 10$, $H = 10$, $Q = 4$, $R = 2$

$J(x(t), u(t)) = 5x^2(10) + \int_0^{10} [2x^2(t) + u^2(t)]\,dt$

(a) Hamiltonian matrix $\mathcal{H} = \left[\begin{array}{c|c} A & -B_u R^{-1} B_u^T \\ \hline -Q & -A^T \end{array}\right] = \begin{bmatrix} 2 & -1/2 \\ -4 & -2 \end{bmatrix}$

(b) $\dot{p}(t) = -4p(t) - 4 + \frac{p^2(t)}{2}$; $p(10) = 10$

(c) $k(t) = R^{-1} B_u^T p(t) = p(t)/2$

There are many ways to solve for $p(t)$

(i) solve $\dot{p}(t) = -4p(t) - 4 + \frac{p^2(t)}{2}$, $p(10) = 10$ numerically using (for example) $p(t-T) = -T\dot{p}(t) + p(t)$ $\left(\because\ \dot{p}(t) \cong \frac{p(t) - p(t-T)}{T}\right)$

(ii) $\int \frac{dp}{\frac{p^2}{2} - 4p - 4} = \int dt$; use partial fractions and integration on L.H.S.

(iii) Use equation (6.16) by noting that you can use $h = \text{expm}(Z * (t_f - t))$ in MATLAB, where Z is the Hamiltonian matrix $\begin{bmatrix} 2 & -1/2 \\ -4 & -2 \end{bmatrix}$. This h is Φ matrix in the last equation on page 186.

(d) steady state gain is $\bar{k} = R^{-1} B_u^T \bar{p} = \bar{p}/2$

where $\frac{\bar{p}^2}{2} - 4\bar{p} - 4 = 0$

using MATLAB or solving the quadratic equation and substituting gives $\bar{k} = 4.45$

(e) closed loop $\dot{x} = Ax - B_u \bar{k} x = -2.45\,x(t)$

$\therefore$ pole $= -2.45$

(f) change Q to $Q = 40$. Pole $p = -4.9$

(g) $J = \frac{1}{2} x^T(0) P x(0)$; $x(0) = 100$

for $Q = 4$, $J = 44495$

for $Q = 40$, $J = 68990$

(6.6) Straight forward (Replace the constant items by the time varying versions in the proof in the book).

(6.7) $x(k+1) = \Phi x(k) + \Gamma u(k)$

$$J(x(k), u(k)) = \frac{1}{2} x^T(k_f) H_d x(k_f) + \frac{1}{2}\sum_{k=0}^{k_f-1}\{x^T(k) Q_d x(k) + u^T(k) R_d u(k)\}$$

(a)

$$J_a(x(k), u(k), p(k+1)) = \frac{1}{2} x^T(k_f) H_d x(k_f) + \frac{1}{2}\sum_{k=0}^{k_f-1}\{x^T(k) Q_d x(k) + u^T(k) R_d u(k) + p^T(k+1)[\Phi x(k) + \Gamma u(k) - x(k+1)]$$

$$\delta J_a = \left[\frac{\partial J_a}{\partial x(k_f)}\right]\delta x(k_f) + \left[\frac{\partial J_a}{\partial u(0)}\right]\delta u(0) + \sum_{k=1}^{k_f-1}\left\{\left[\frac{\partial J_a}{\partial x(k)}\right]\delta x(k) + \left[\frac{\partial J_a}{\partial u(k)}\right]\delta u(k) + \delta p^T(k)\left[\frac{\partial J_a}{\partial p(k)}\right]\right\}$$

Before we apply this let us re-write J_a as

$$J_a(x(k), u(k), p(k+1)) = \frac{1}{2} x^T(k_f) H_d x(k_f) - p^T(k_f) x(k_f) + x^T(0) Q_d x(0) + u^T(0) R_d u(0) + p^T(1)[\Phi x(0) + \Gamma u(0)] + \sum_{k=1}^{k_f-1}\{x^T(k) Q_d x(k) + u^T(k) R_d u(k) + p^T(k+1)[\Phi x(k) + \Gamma u(k)]\}$$

$$\therefore \delta J_a = [x^T(k_f) H_d - p^T(k_f)]\delta x(k_f) + [u^T(0) R_d + p^T(1)\Gamma]\delta u(0) + \sum_{k=1}^{k_f-1}[x^T(k) Q_d + p^T(k+1)\Phi - p^T(k)]\delta x(k) + [u^T(k) R_d + p^T(k+1)\Gamma]\delta u(k) + \delta p^T(k)[x(k) - \Phi x(k-1) + \Gamma u(k-1)]$$

$\therefore$ Necessary conditions are:

$$\boxed{\begin{aligned} p^T(k_f) &= x^T(k_f) H_d \\ p(k) &= Q_d x(k) + \Phi^T p(k+1) \\ x(k+1) &= \Phi x(k) + \Gamma u(k) \\ u(k) &= -R_d^{-1}\Gamma^T p(k+1) \end{aligned}}$$

or $p(k_f) = H_d x(k_f)$

(b) $p(k) = P(k)x(k)$

$u(k) = -R_d^{-1}\Gamma^T p(k+1)$; ∴ we need $p(k+1)$

$p(k+1) = P(k+1)x(k+1) = P(k+1)[\Phi x(k) + \Gamma u(k)]$

$u(k) = -R_d^{-1}\Gamma^T P(k+1)[\Phi x(k) + \Gamma u(k)]$

or $[I + R_d^{-1}\Gamma^k P(k+1)\Gamma]u(k) = -R_d^{-1}\Gamma^T P(k+1)\Phi x(k)$

Premultiplying by Rd and then solve for u(t) gives

∴ $u(k) = -[R_d + \Gamma^T P(k+1)\Gamma]^{-1}\Gamma^T P(k+1)\Phi x(k)$

(c) $x(k+1) = \Phi x(k) + \Gamma u(k) = \Phi x(k) - \Gamma R_d^{-1}\Gamma^T P(k+1)x(k+1)$; moving terms

we get,

$x(k+1) = [I + \Gamma R_d^{-1}\Gamma^T P(k+1)]^{-1}\Phi x(k)$ — 1

we know that $p(k) = Q_d x(k) + \Phi^T p(k+1)$

and $p(k) = P(k)x(k)$

∴ $P(k)x(k) = Q_d x(k) + \Phi^T P(k+1)x(k+1)$ — 2

using 1 in 2 gives

$P(k) = \Phi^T P(k+1)[I + \Gamma R_d^{-1}\Gamma^T P(k+1)]^{-1}\Phi + Q_d$

using the matrix inversion lemma (page 437) gives

$P(k) = \Phi^T P(k+1)\Phi + Q_d - \Phi^T P(k+1)\Gamma[R_d + \Gamma^T P(k+1)\Gamma]^{-1}\Gamma^T P(k+1)\Phi$

(d) $p(k_f) = P(k_f)x(k_f) = H_d x(k_f)$

∴ $\boxed{P(k_f) = H_d}$

6.9 $\dot{x}(t) = Ax(t) + Bu(t)$

$J(x(t), u(t)) = \frac{1}{2}x^T(t_f)Hx(t_f) + \frac{1}{2}\int_0^{t_f}\{[x(t) - r(t)]^T Q[x(t) - r(t)] + u^T(t)Ru(t)\}dt$

$$J_a = J + \int_0^{t_f} p^T(t)\left[Ax(t) + Bu(t) - \dot{x}(t)\right]dt$$

$$\delta J_a = x^T(t_f)H\delta x(t_f) + \int_0^{t_f} \Big[\left(x(t) - r(t)\right)^T Q + p^T(t)A\Big]\delta x(t) + \left[u^T(t)R + p^T(t)B\right]\delta u(t)$$

$$+ \delta p^T(t)\left[Ax(t) + Bu(t) - \dot{x}\right] - p^T(t)\,\delta\dot{x}(t)\Big]dt = 0$$

Now, $$\int_0^{t_f} p^T(t)\,\delta\dot{x}(t)\,dt = p^T(t_f)\delta x(t_f) - p^T(0)\cancel{\delta x(0)}^{\,0} - \int_0^{t_f}\dot{p}^T(t)\,\delta x(t)\,dt$$

$$\therefore \quad p^T(t_f) = x^T(t_f)H$$

$$\dot{p}^T(t) = -\left(x(t) - r(t)\right)^T Q - p^T(t)A$$

$$u(t) = -R^{-1}B^T p(t)$$

$$\begin{bmatrix}\dot{x}(t)\\ \dot{p}(t)\end{bmatrix} = \begin{bmatrix}A & -BR^{-1}B^T\\ -Q & -A^T\end{bmatrix}\begin{bmatrix}x(t)\\ p(t)\end{bmatrix} + \begin{bmatrix}0\\ Q\end{bmatrix} r(t)$$

$$\begin{bmatrix}x(t_f)\\ p(t_f)\end{bmatrix} = e^{A(t_f - t)}\begin{bmatrix}x(t)\\ p(t)\end{bmatrix} + \int_t^{t_f} e^{A(t_f-\tau)}\begin{bmatrix}0\\ Q\end{bmatrix} r(\tau)\,d\tau$$

$$\begin{bmatrix}x(t_f)\\ p(t_f)\end{bmatrix} = \left[\begin{array}{c|c}\Phi_{11}(t_f-t) & \Phi_{12}(t_f-t)\\ \hline \Phi_{21}(t_f-t) & \Phi_{22}(t_f-t)\end{array}\right]\begin{bmatrix}x(t)\\ p(t)\end{bmatrix} + \begin{bmatrix}\overbrace{Q\int_t^{t_f}\Phi_{12}(t_f-\tau)r(\tau)\,d\tau}^{w_1}\\ \underbrace{Q\int_t^{t_f}\Phi_{22}(t_f-\tau)r(\tau)\,d\tau}_{w_2}\end{bmatrix}$$

$$p(t_f) = Hx(t_f)$$

$$Hx(t_f) = \Phi_{21}(t_f-t)x(t) + \Phi_{22}(t_f-t)p(t) + w_2$$

$$H\left[\Phi_{11}(t_f-t)x(t) + \Phi_{12}(t_f-t)p(t)\right] + HQ\int_t^{t_f}\Phi_{12}(t_f-\tau)\,r(\tau)\,d\tau$$

$$= \Phi_{21}(t_f-t)p(t) + Q\int_t^{t_f}\Phi_{22}(t_f-\tau)\,r(\tau)\,d\tau$$

$$\left[H\Phi_{11}(t_f-t) - \Phi_{21}(t_f-t)\right]x(t) + \left[H\Phi_{12}(t_f-t) - \Phi_{22}(t_f-t)\right]p(t)$$

$$\cancel{s(t_f)} \qquad \alpha\begin{cases} + HQ\displaystyle\int_t^{t_f}\Phi_{12}(t_f-\tau)\,r(\tau)\,d\tau \\ - Q\displaystyle\int_t^{t_f}\Phi_{22}(t_f-\tau)\,r(\tau)\,d\tau \end{cases}$$

$$p(t) = \left[H\Phi_{12}(t_f-t) - \Phi_{22}(t_f-t)\right]^{-1}\left[H\Phi_{11}(t_f-t) - \Phi_{21}(t_f-t)\right]x(t)$$

$$+\left[H\Phi_{12}(t_f-t) - \Phi_{22}(t_f-t)\right]^{-1}\left[HQ\int_t^{t_f}\Phi_{12}(t_f-\tau)r(\tau)\,d\tau - Q\int_t^{t_f}\Phi_{22}(t_f-\tau)r(\tau)\,d\tau\right]$$

$\therefore p(t) = P(t)x(t) + s(t)$

we can see that $s(t_f) = 0$ ~~and~~

$p(t_f) = Hx(t_f) = P(t_f)x(t_f) + s(t_f)$

$\therefore \boxed{P(t_f) = H}$ and $\boxed{s(t_f) = 0}$

$\dot p(t) = \dot P(t)x(t) + \dot s(t) + P(t)\dot x(t)$

$$-Qx(t) - A^TP(t)x(t) + Qr(t) - A^Ts(t) = P(t)\left[Ax(t) - BR^{-1}B^TP(t)x(t) - BR^{-1}B^Ts(t)\right] + \dot s + \dot P(t)x(t)$$

$$A^Ts(t) + \dot s(t) + P(t)BR^{-1}B^TP(t)s(t) + \dot P(t)x(t) = \left[-P(t)A - A^TP(t) - Q + P(t)BR^{-1}B^TP(t)\right]x(t) + Qr(t)$$

If $r(t) \equiv 0$ then $s(t) \equiv 0$

$\therefore$ we get

$$\boxed{\dot P(t) = -P(t)A - A^TP(t) - Q + P(t)BR^{-1}B^TP(t)}, \quad \boxed{P(t_f) = H}$$

$$\cancel{\boxed{\dot s(t) = Qr(t)}}, \quad \boxed{s(t_f) = 0}, \quad \boxed{\dot s(t) = -A^Ts(t) - P(t)BR^{-1}B^Ts(t) + Qr(t)}$$

and $u(t) = -R^{-1}B^T\left[P(t)x(t) + s(t)\right]$

$$= \underbrace{-K(t)x(t)}_{\text{feedback term}} \underbrace{-R^{-1}B^Ts(t)}_{\text{feedforward term}}$$

Feedforward in general is sensitive to modeling uncertainties.