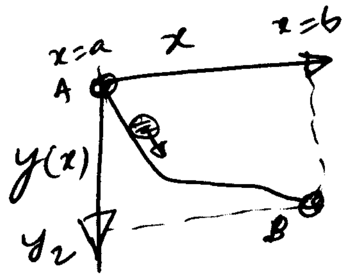
Calculus of Variations Example



min time for ball to go from A to B

given $y(x_1) = 0$, $y(x_2) = y_2$

$$dt = \frac{ds}{v} \quad ; \; ds = \text{differential distance}$$

$$ds = \sqrt{dx^2 + dy^2}$$

$$\therefore \quad T = \int_{x=a}^{x=b} \frac{ds}{v} = \int_a^b \frac{\sqrt{1+y'^2}}{v}\, dx$$

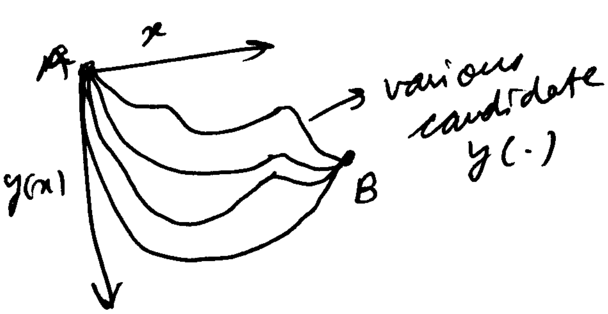
$$\frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 = mg(y_1 - y_2)$$

Taking $v_1 = 0$, $y_1 = 0$ then

$$v = \sqrt{2gy}$$

$$\therefore \quad T = \frac{1}{\sqrt{2g}} \int_a^b \frac{\sqrt{1+y'^2}}{\sqrt{y}}\, dx$$

find $y(x)$, $x \in [a,b]$ to minimise T.



$$J(y(\cdot)) = \int_a^b f(x, y(x), y'(x))\, dx$$

$$y(a) = y_1 \; ; \; y(b) = y_2$$

Calculus of Variations (Simplified Version)

min $\quad J(y(\cdot)) = \int_a^b f(x, y(x), y'(x))\,dx \quad ; \quad y(a) = y_1 \; ; \; y(b) = y_2$

$\cancel{\hat{y}}$ Assume $y^*(\cdot)$ is a local minimizer

$$\hat{y}(x) = y^*(x) + \epsilon\,\eta(x)$$

$$F(\epsilon) = \int_a^b f(x, \hat{y}(x), \hat{y}'(x))\,dx$$

$\boxed{\text{NOTE: Consider } f(x, y(x), y'(x)) \text{ as } f(x, y, z)}$

$$F'(0) = \int_a^b (f_y \cdot \eta(x) + f_z \cdot \eta'(x))\,dx$$

$$= \int_a^b (f_y \cdot \eta(x) - \frac{d}{dx} f_z)\,dx + f_z \cdot \cancel{\eta(x)}^{\,0}\Big|_a^b \qquad (\because \eta(a) = \eta(b) = 0)$$

$\therefore$ Euler-Lagrange necessary condition is

$$\boxed{\frac{d}{dx} f_z(x, y^*(x), y^{*\prime}(x)) = f_y(x, y^*(x), y^{*\prime}(x)) \qquad \forall x \in [a, b]}$$

If the problem has one fixed point $y(a) = y_1$ and other end free then at that end we will have to satisfy

$$\boxed{f_z\Big|_b = 0}$$

Two Independent Variable Problem

$$J = \iint_D f(x, y, \omega, \omega_x, \omega_y)\,dx\,dy$$

points given on the boundary ∂D.

$$W(x, y) = \omega(x, y) + \epsilon\,\eta(x, y)$$

$$\therefore \; \eta(x, y) = 0 \quad \forall (x, y) \in \partial D$$

$$F(\epsilon) = \iint_D f(x, y, W, W_x, W_y)\,dx\,dy$$

solve $\; F'(\epsilon) = 0 \,\big|\,$ at $\epsilon = 0$

$$F'(0) = \iint_D (f_\omega \eta + f_{\omega_x}\eta_x + f_{\omega_y}\eta_y)\,dx\,dy = 0 \quad \text{— (A)}$$

<u>Green's theorem:</u> $$\boxed{\iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\,dy = \int_C (P\,dy - Q\,dx)}$$

Take $P = \eta G$ and $Q = \eta F$, then

$$\iint_D \left(G\frac{\partial \eta}{\partial x} + F\frac{\partial \eta}{\partial y}\right) dx\,dy = -\iint_D \eta\left(\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y}\right) dx\,dy + \int_C \eta\,(G\,dy - F\,dx) \quad \text{— (B)}$$

Applying (B) to (A) where $G = f_{\omega_x}$; $F = f_{\omega_y}$

we get

$$F'(0) = \iint_D \eta\left[f_\omega - \frac{\partial}{\partial x} f_{\omega_x} - \frac{\partial}{\partial y} f_{\omega_y}\right] dx\,dy + \underbrace{\int_C \eta\,(f_{\omega_y}\,dy - f_{\omega_x}\,dx)}_{0} = 0$$

$$\therefore \quad \boxed{f_\omega = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \omega_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \omega_y}\right)}$$

<u>Linear Transformation</u>

A mapping T of a vector space X into a vector space Y is called a linear transformation ($T \in L(X,Y)$) if

(i) $T(x+y) = T(x) + T(y) \quad \forall\, x, y \in X$ and

(ii) $T(\alpha x) = \alpha T(x) \quad \forall\, x \in X$ and $\alpha \in F$ (field, e.g real number or complex number)

<u>Adjoint Transformation</u>

For each $G \in L(X,X)$, $\exists$ a unique $G^* \in L(X,X)$ s.t $(x, G^* y) = (Gx, y)\ \forall\, x, y \in X$ called the adjoint of G,

<u>note</u>: (x,y) is the inner product of two vectors x and y

e.g For G a real matrix, $G^* = G^T$

" " " complex " $G^* = \hat{G}^T$

$\hat{G}$ = complex conjugate of G.

Image Restoration

$$u_0 = R u + \eta$$

u → noiseless image; u_0 ← noisy image; R → linear operator representing blur (convolution); η ← noise

The image can be written as a one dimensional vector.

$u, u_0 \in R^M$; $R \in M \times M$

$$\boxed{\inf_u \int_\Omega (u_0 - Ru)^T (u_0 - Ru)\, dx} \quad -(3.2)$$

$$f = (u_0 - Ru)^T (u_0 - Ru) = u_0^T u_0 - u_0^T R u - u^T R^T u_0 + u^T R^T R u$$

Applying Euler-Lagrange here gives

$f_u = 0$

$$f_u = -R^T u_0 - R^T u_0 + 2 R^T R u = 0$$

$$\boxed{R^T u_0 - R^T R u = 0} \quad -(3.3)$$

Ill-posed problem if $R^T R$ singular.

Note:
$\nabla_x (x^T A y) = Ay$
$\nabla_x (y^T A x) = A^T y$
$\nabla_x (x^T A x) = 2Ax$

Regularization

$$\boxed{F(u) = \int_\Omega (u_0 - Ru)^T (u_0 - Ru) + \lambda (\nabla u^T \nabla u)\, dx} \quad -(3.4)$$

Here compared to $f(x, y, z)$; y is u and ∇u is z

Euler Lagrange

$$\frac{d}{dx} f_z = f_u$$

Note:
$f_z = 2\nabla u = 2\frac{\partial u}{\partial x}$
$\frac{d}{dx} f_z = 2\frac{\partial^2 u}{\partial x^2} = 2\Delta u$

$$\Rightarrow \boxed{R^T R u - R u_0 - \lambda \Delta u = 0} \quad -(3.5)$$

Boundary condition $\frac{\partial u}{\partial N} = 0$ on $\partial\Omega$ (N is the outward normal to $\partial\Omega$)

Effect of Δu in (3.5) smooths out edges (Figure 3.3)

Problem is the L_2 norm on ∇u i.e the $\nabla u^T \nabla u$ term using 2 independent variables version of Euler-Lagrange

$$\nabla u^T \nabla u = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \quad ; u = u(x,y)$$

$$f_u = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial u_y}\right)$$

$$R^T R u - R u_0 - \lambda \Delta u = 0$$

where $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

Take L_1 norm based problem

$$\boxed{E(u) = \frac{1}{2}\int_\Omega |u_0 - Ru|^2 dx + \lambda \int_\Omega \phi(|\nabla u|)dx} \quad -(3.6)$$

Euler Lagrange gives

$$\boxed{R^T R u - \lambda \operatorname{div}\left(\frac{\phi'(|\nabla u|)}{|\nabla u|}\nabla u\right) = R^T u_0} \quad -(3.7)$$

note: given a vector $V = (V_1 \; V_2 \; V_3)$

$$\operatorname{div} V = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$