INTRODUCTION

DYNAMIC SYSTEM MODEL

$\dot{x} = f\,[t, x(t), u(t)], \ \forall t \geq 0,$ (First order vector differential equation)

$t \in R^+$ (time), $x(t) \in R^n$, $u(t) \in R^m$

$f: R^+ \times R^n \times R^m \to R^n$

$x(t)$: state
$u(t)$: input or control

$y = h(t, x, u)$
$y \in R^q$ (output)

$$\dot{x}_1 = f_1(t, x_1, x_2, \dots, x_n, u_1, \dots u_m)$$
$$\dot{x}_2 = f_2(t, x_1, x_2, \dots, x_n, u_1, \dots u_n)$$
$$\vdots$$
$$\dot{x}_n = f_n(t, x_1, x_2, \dots, x_n, u_1, \dots u_m)$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \quad f(t, x, u) = \begin{bmatrix} f_1(t, x, u) \\ f_2(t, x, u) \\ \vdots \\ f_n(t, x, u) \end{bmatrix}$$

- Unforced Dynamic System
  $\dot{x} = f(t, x)$ (independent of u)
- forced Dynamic System
  $\dot{x} = f(t, x, u)$

NOTE: If $u = \gamma(x)$ function of time yields an unforced system

- If f independent of t, then system is autonomous (time-invariant)
  otherwise nonautonomous (time-varying)

- Analysis for a given $u(t)$ analyze $x(t)$ behavior
- Design find out u to have some specified properties of $x(t)$.

EQUILIBRIUM $x_0 \in R^n$ is an equilibrium of the unforced system

$$\dot{x} = f[t, x(t)] \quad \text{if} \quad f[t, x_0] = 0 \quad \forall t \geq 0$$

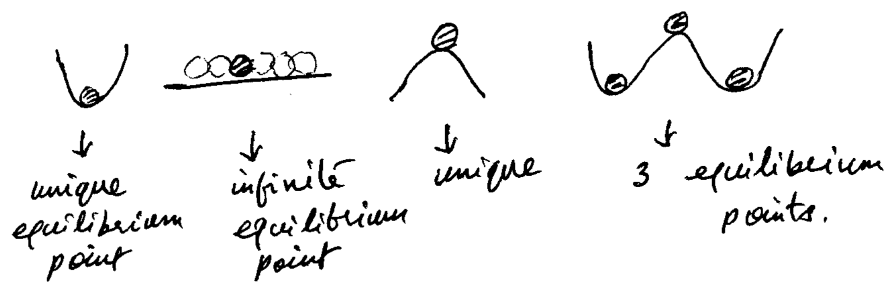
If x_0 is an equilibrium then

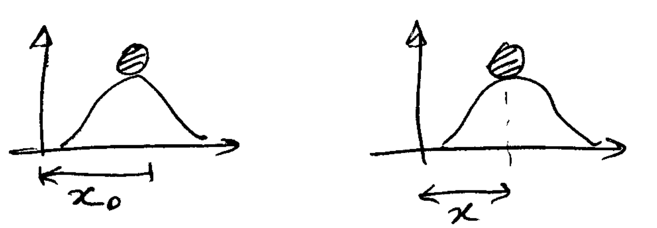
$$\dot{x}(t) = f[t, x(t)] \;, \forall t \geq t_0 \;; \; x(t_0) = x_0$$

has the unique solution

$$x(t) = x_0 \;, \quad \forall t \geq t_0$$

i.e if system starts in equilibrium, it should remain in that state thereafter.





$t = 0$
$x(0) = x_0$

$x(t) = x_0$
$\forall t \geq 0$

NONLINEARITY

- **Linearity** A function f is linear if:

  (a) $f(u_1+u_2) = f(u_1) + f(u_2)$ for any u_1 and u_2 in the domain of f

  (b) $f(\alpha u) = \alpha f(u)$ for any u in the domain of f and for any real number α.

  If f is not linear, then f is nonlinear.

- **Nonlinearities**
  - inherent (system dynamics)
  - intentional or artificial (introduced by u)

  - continuous
  - discontinuous (hard nonlinearity)

  - when nonlinearity is continuous, then the behavior of the system can be approximated by a linear system in a small range

Linear System (LTI, linear time invariant system)

$$\dot{x} = Ax$$

A is the system matrix

- a linear system has a unique equilibrium point if A is nonsingular
- the equilibrium point is stable if all eigenvalues of A have negative real parts, regardless of initial conditions.
- the transient response consists of natural modes
- the general solution can be solved analytically
- for a forced system

  $$\dot{x} = Ax + Bu$$

  - the system satisfies the principle of superposition
  - sinusoidal input leads to sinusoidal output of the same frequency
  - we can use Laplace techniques
- If $\dot{x} = Ax$ is asymptotically stable, then for $\dot{x} = Ax + Bu$ bounded-input bounded-output (BIBO) stability is implied

Nonlinear Systems

1) Multiple Equilibrium Points:

Consider example

$\dot{x} = -x + x^2$ — ① with initial condition $x(0) = x_0$

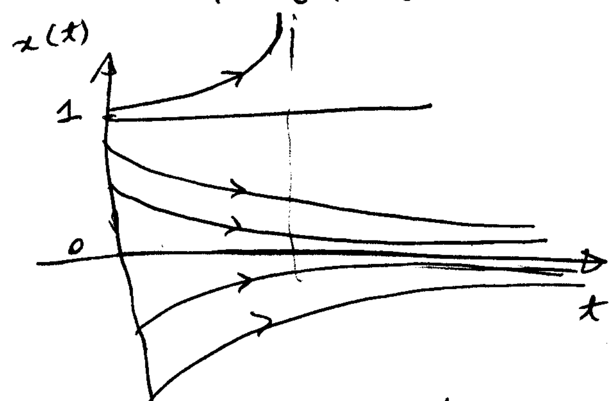
equilibrium points are given by solving

$-x + x^2 = 0$

$x(x-1) = 0$ i.e $x = 0$ and 1

By integrating ① we get

$$x(t) = \frac{x_0 e^{-t}}{1 - x_0 + x_0 e^{-t}}$$

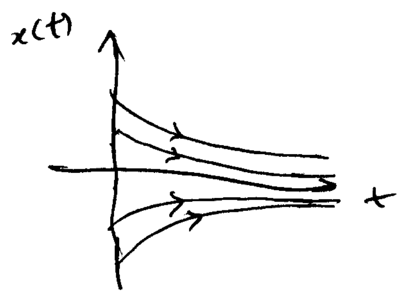


system trajectory stable or unstable depending on initial conditions.

study the linearization

$\dot{x} = -x$, $x(0) = x_0$

$x(t) = x_0 e^{-t}$



single equilibrium at 0 stable for any initial value

2) Finite escape time for nonlinear systems

- unstable ~~[illegible]~~ linear systems can have the state go to infinity as time → infinity
- nonlinear systems can have state go to infinity in finite time (as shown in example above)

3) $\dot{x} = xu$

for $u = -1$, system is stable converging to 0

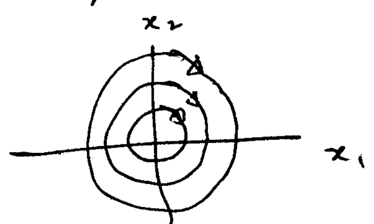
for $u = 1$, " " unstable

4) Look at $\ddot{y} = -\omega^2 y$, $y(0) = y_0$, $\dot{y}(0) = \dot{y}_0$

$y(t) = A \sin\omega t + B \cos\omega t$; A and B depending on y_0 and $\dot{y}_0$

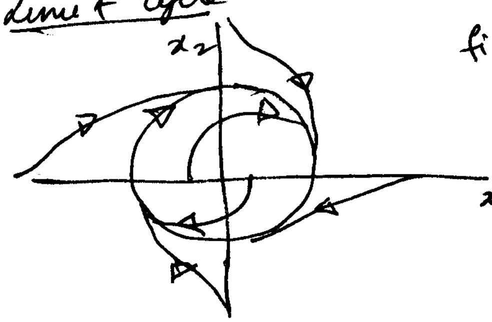
$x_1 = y$
$x_2 = \dot{y}$

Laplace $s^2 + \omega^2 = 0 \Rightarrow \lambda = +j\omega, -j\omega$



Amplitude of oscillation depends on y_0 and $\dot{y}_0$
i.e on $[x_1\ x_2]^T$
i.e on x

<u>Limit Cycle</u> in a nonlinear system



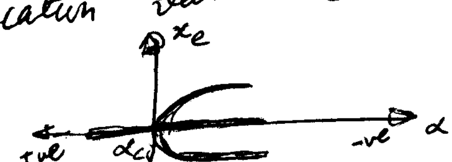
fixed amplitude and fixed frequency oscillation independent of initial conditions.

5) <u>Bifurcations</u>: As parameters of nonlinear dynamic systems are changed, the stability and even the number of equilibrium points can change. The values of these parameters at which the qualitative behavior of the system changes are called <u>critical or bifurcation values.</u>

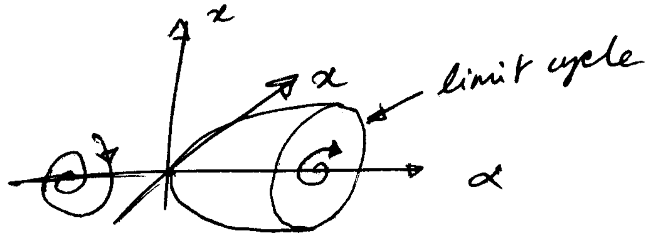
a) Consider undamped Duffing equation

$$\ddot{x} + \alpha x + x^3 = 0$$

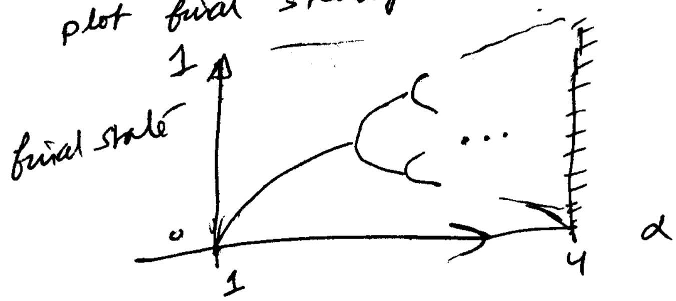
As α varies from positive to negative, one equilibrium point at $x_e = 0$ split into three at $x_e = 0, \sqrt{\alpha}, -\sqrt{\alpha}$. $\therefore$ $\alpha = 0$ is a critical bifurcation value. (Pitchfork bifurcation)



b) <u>Hopf Bifurcation</u>: Emergence of limit cycles as parameters are changed. A pair of complex ~~eigenvalues~~ conjugate eigenvalues cross from left half plane to right and produce a limit cycle.



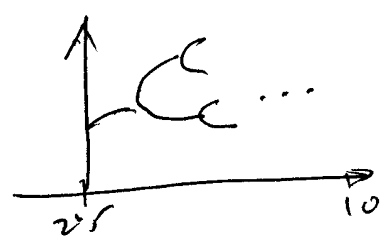
c) Consider $x(k+1) = \alpha x(k)(1 - x(k))$ [Quadratic Iterator]
plot final steady state value against α.



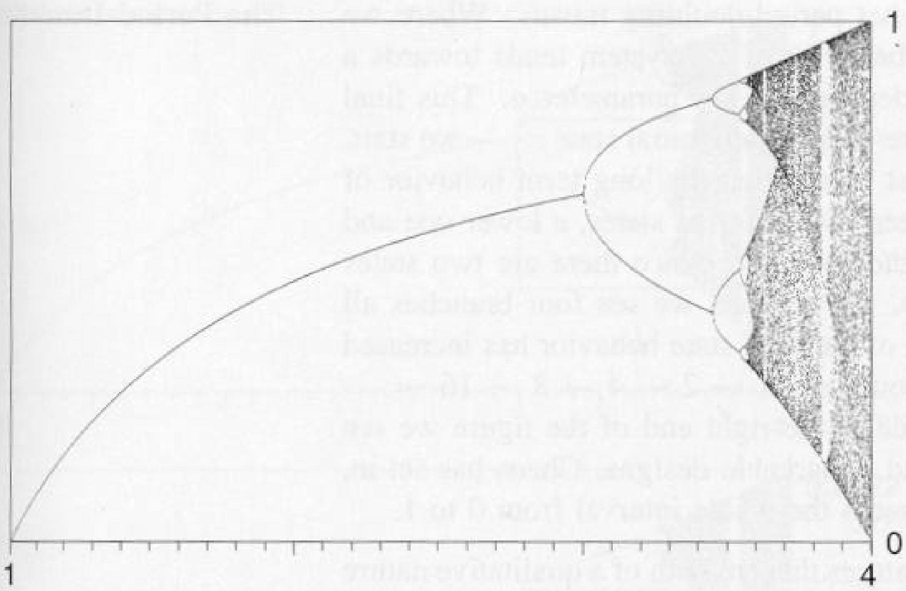
Consider (Rössler Attractor)

$$\dot{x} = -y + z$$
$$\dot{y} = x + \alpha y$$
$$\dot{z} = \beta + xz - \gamma z$$

vary parameter γ and observe bifurcation in final s.s value



6) <u>Chaos</u>: For stable linear systems, small differences in initial conditions can cause only small differences in output. In chaotic nonlinear systems we see a deterministic system producing unpredictability of output.
Quadratic Iterator at $\alpha = 4$ shows chaos



Final-State Diagram

Final-state diagram for the quadratic iterator and parameter a between 1 and 4.

Error Development

The quadratic iterator $x \to 4x(1-x)$ applied to two initial values differing by 10^{-6} (top and center) and the (absolute) difference of the two signals (bottom).

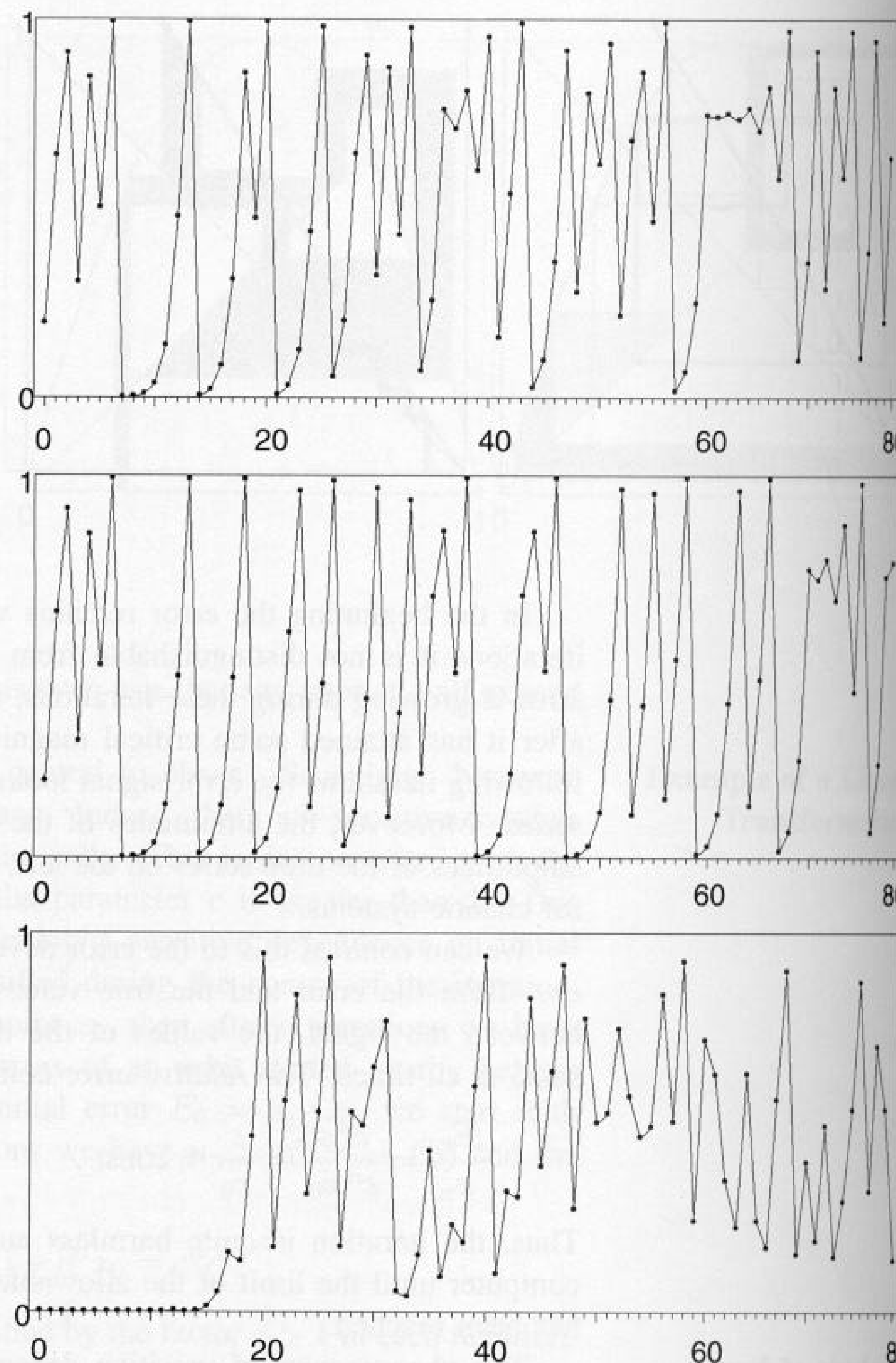


Figure 10.7

single order for discrete and 2nd order forced or third order unforced for continuous systems. Example of 2nd order forced producing chaos is

$$\ddot{x} + 0.05\dot{x} + x^3 = 7.5 \cos t$$

7) Subharmonic, harmonic or almost-periodic oscillations:
A stable linear system produces a sinusoidal output of a frequency equal to the frequency of the sinusoidal input. Nonlinear system output can have a multiple or submultiple frequency of the input or almost periodic oscillations (sum of frequencies not multiples of each other).

8) Multiple modes of behavior: As parameters or inputs are changed (even smoothly) the behavior can change from one mode to another, such as limit cycles, subharmonic, etc.

Homework

Ex. 1.1, 1.2, 1.3, 1.4, 1.5, 1.7

P7: Use MATLAB to find the state value after 100 steps for $\alpha = 1, 2, 3, 4$. For each value of α, run the simulation at least 20 times each ~~test~~ a different starting value for $x(0)$ in

$$x(k+1) = \alpha\, x(k)(1 - x(k)).$$

P8: For

$$\dot{x} = -\operatorname{sign} x(t), \ \forall t \geq 0, \ x(0) = 0$$

$$\text{where } \operatorname{sign} x = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Find the equilibrium points.

SECOND ORDER SYSTEMS

$\dot{x}_1 = f_1(x_1, x_2)$
$\dot{x}_2 = f_2(x_1, x_2)$

- State Plane = two dimensional plane with x_1 on x-axis and x_2 on y axis

- state plane Plot or trajectory: If $x_1(t)$ and $x_2(t)$ solution, then plot of $x_2(t)$ versus $x_1(t)$ as t varies over R^+

If $\dot{x}_1 = x_2$, then state plane is called phase plane, and state plane plot or trajectory called phase " " or ".

- Smooth function: A function $f: R^2 \to R$ is smooth if $f(x_1, x_2)$ has continuous partial derivatives of all orders with respect to all possible combinations of x_1 and x_2.

- vector field: A function $f: R^2 \to R^2$ is called a vector field if both its components are smooth functions. Also called velocity vector field.

- vector $x \in R^2$ is an equilibrium of a vector field f if $f(x) = 0$. Also called singular point.

- If $f(x) \neq 0$ then the direction of vector field f at the point x is defined by
$\theta_f(x) = A\tan[f_1(x), f_2(x)]$
(NOTE: $A\tan[a, b]$ is a unique number θ, such that $\theta \in [0, 2\pi)$
$\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$

Example:

$\ddot{x} + x = 0$, given $x(0) = x_0$, $\dot{x}(0) = 0$

solution $x(t) = x_0 \cos t$

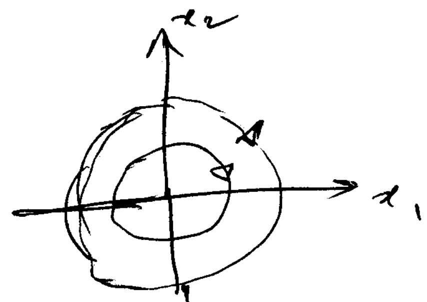
$\dot{x}(t) = -x_0 \sin t$

eliminating t, we get

$x^2 + \dot{x}^2 = x_0^2$

define $x_1 = x$
$x_2 = \dot{x}$ also $\Rightarrow$ $x_1^2 + x_2^2 = x_0^2$

$\Rightarrow \dot{x}_1 = x_2$, $\dot{x}_2 = -x_1$



Constructing Phase Portraits

- Computer Generated
- others
  - Analytic method
  - Isocline
  - delta, Lienard, Pell etc. methods

Analytic method

1) Eliminate time using solutions of $x_1(t)$ and $x_2(t)$ (shown in example above)

$\Rightarrow$ use $\dfrac{dx_2}{dx_1} = \dfrac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$

$\dot{x}_1 = x_2$

$\dot{x}_2 = -x_1$

$\therefore \dfrac{dx_2}{dx_1} = -\dfrac{x_1}{x_2}$

$\Rightarrow \dot{x}\dfrac{d\dot{x}}{dx} + x = 0$

Integration yields
$\dot{x}^2 + x^2 = x_0^2$

The Method of Isoclines

note $\dfrac{dx_2}{dx_1} = \dfrac{f_2(x_1,x_2)}{f_1(x_1,x_2)} = \alpha$ (solve for constant α)

gives equation $f_2 = \alpha f_1$

This is an equation for constant α slope.

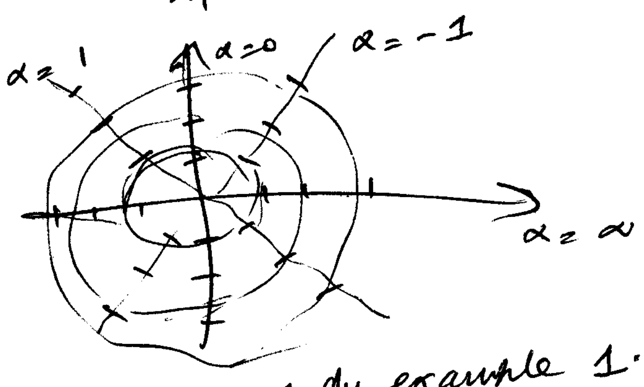
Take different values of α, and solve different equations.

example.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1$$

$$\frac{dx_2}{dx_1} = \frac{-x_1}{x_2} = \alpha$$

$$x_1 + \alpha x_2 = 0$$



(study example 1.1)

OBTAINING TIME FROM PHASE PORTRAITS

$$\frac{dx}{dt} = \dot{x} \Rightarrow dt = \frac{dx}{\dot{x}}$$

$$t - t_0 = \int_{t_0}^{t} dt = \int_{x_0}^{x} \frac{dx}{\dot{x}}$$

i.e if one use $\frac{1}{\dot{x}}$ and x as the coordinates, then the area under the curve gives time elapsed.

Phase Plane Analysis of Linear System

Take $\ddot{x} + a\dot{x} + bx = 0$

use Laplace

$$s^2 + as + b = (s-\lambda_1)(s-\lambda_2) = 0$$

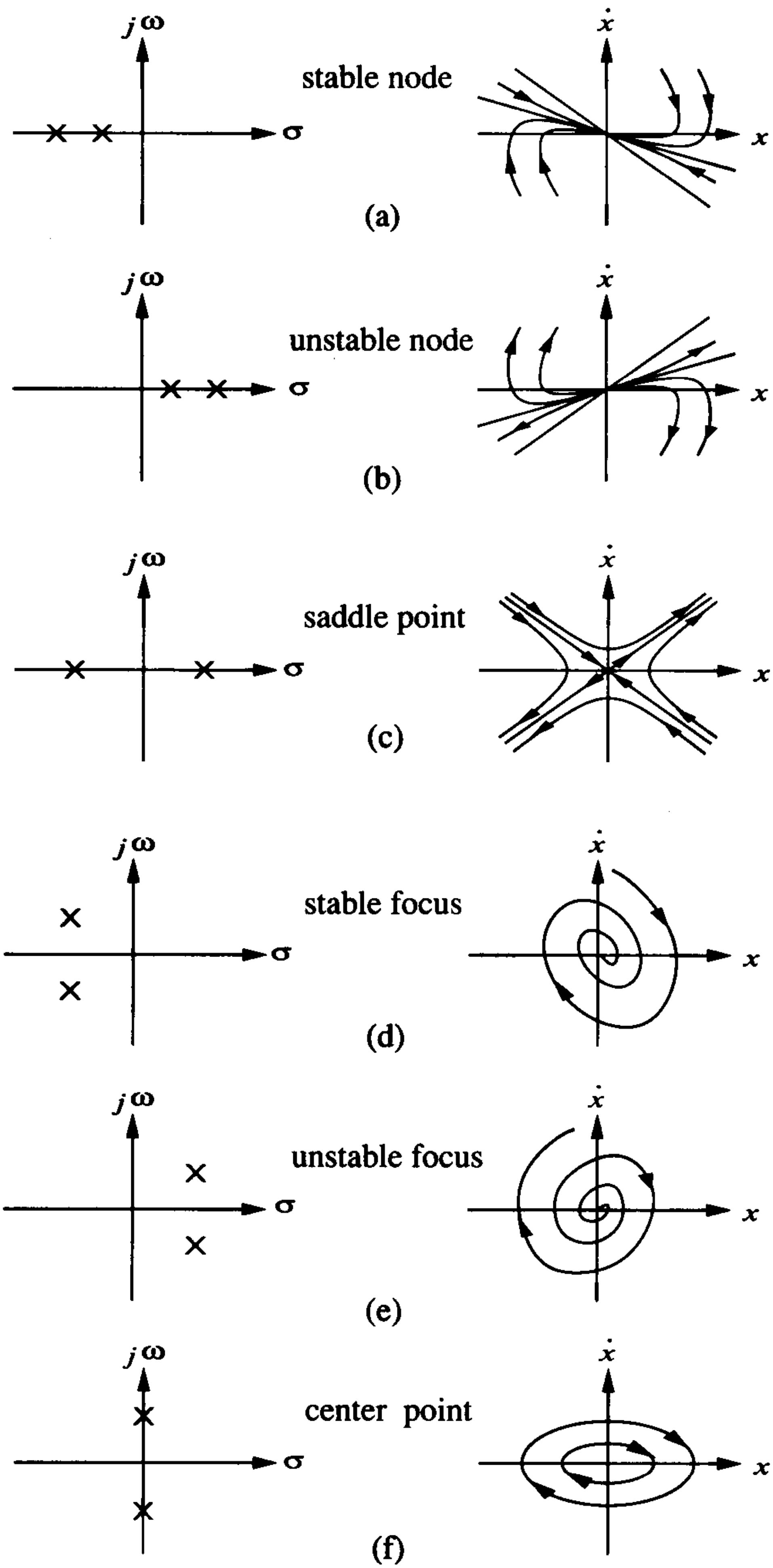
$$\lambda_1 = (-a + \sqrt{a^2-4b})/2$$

$$\lambda_2 = (-a - \sqrt{a^2-4b})/2$$

<u>solution</u> $x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}$ for $\lambda_1 \neq \lambda_2$

$x(t) = k_1 e^{\lambda_1 t} + k_2 t\, e^{\lambda_1 t}$ for $\lambda_1 = \lambda_2$

If $b \neq 0$, only one equilibrium point (at $x = 0, \dot{x} = 0$)

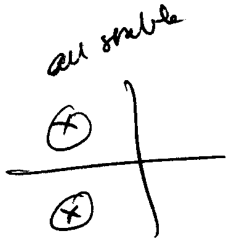


stable node
(a)
unstable node
(b)
saddle point
(c)
stable focus
(d)
unstable focus
(e)
center point
(f)

LOCAL BEHAVIOR OF NONLINEAR SYSTEMS

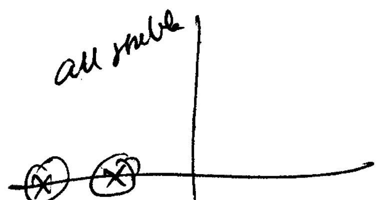
suppose A has distinct eigenvalues.. Consider $A+\Delta A$, elements of ΔA are arbitrarily small in magnitude.

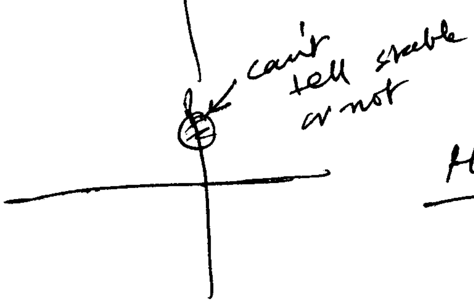
Eigenvalues of matrices depend continuously on their parameters.



∴ make a small change in ΔA causes small change in λ.

Given any positive number ϵ, $\exists$ a corresponding positive number δ, such that if the magnitude of each element of ΔA is less than δ, then the eigenvalues of $A+\Delta A$ will lie in a ball centered at eigenvalues of A with a radius of ϵ.





Hyperbolic Equilibrium Point

$x=0$ is a hyperbolic equilibrium point of $\dot{x}=Ax$, if A has no eigenvalues with zero real part.

∴ nonlinear systems whose linearization has hyperbolic eq. point show the linearized local behavior. (other ones, one can't predict)

∴

Equilib. of linearized system	Equilib. of nonlinear
stable node	stable node
unstable node	unstable node
saddle	saddle
stable focus	stable focus
unstable focus	unstable focus
Center	?

given $\dot{x} = f(x)$

linearization yields (at $x=0$)

<u>note</u>: Here $x=0$ taken as equilibrium.
If $x=0$ not an equilibrium, define change of coordinates to accomplish that.

$$\dot{x} = \left(\frac{\partial f}{\partial x}\right)_{x=0} + f_{h.o.t}(x)$$

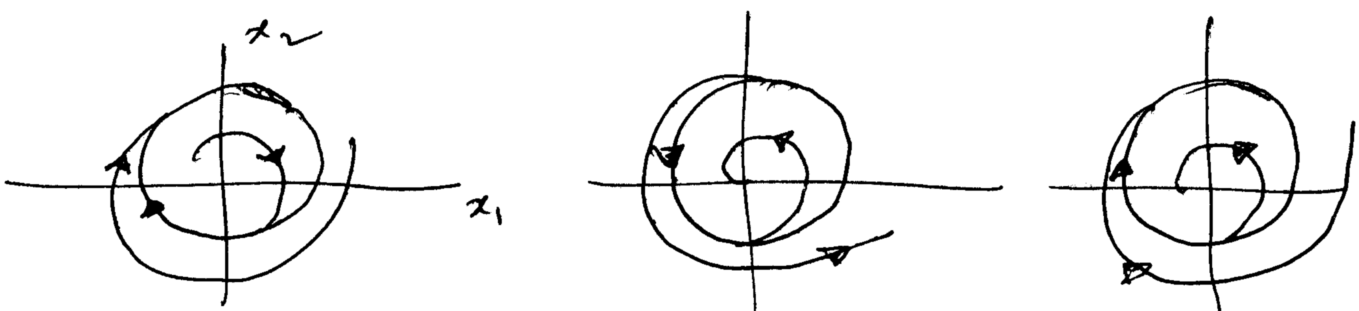
higher order terms

$$\therefore A = \left(\frac{\partial f}{\partial x}\right)_{x=0}$$

for $\dot{x} = Ax$

LIMIT CYCLE

Isolated closed curve in the phase plane.
(Compare it to centers which are not isolated)



Stable	Unstable	Semi stable
all trajectories converge	diverge	some converge some diverge

example

$$\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1)$$

$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1)$$

use $r = (x_1^2 + x_2^2)^{1/2}$ $\quad \theta = \tan^{-1}(x_2/x_1)$

$$\therefore \frac{dr}{dt} = -r(r^2 - 1)$$

$$\frac{d\theta}{dt} = -1$$

notice at $r = 1$, $\dot{r} = 1$

for $r < 1$, $\dot{r} > 0$

for $r > 1$, $\dot{r} < 0$

$\therefore$ stable limit cycle ~~for~~ $r = 1$.

(Also see analytical solution

$$r(t) = \frac{1}{(1 + c_0 e^{-2t})^{1/2}}$$

$$\cancel{\theta(t) = \frac{1}{r_0^2}}$$

$$\theta(t) = \theta_0 - t$$

$$c_0 = \frac{1}{r_0^2} - 1$$

☆ $\dot{x} = f(x)$

+ve semi orbit through $y \Rightarrow \gamma^+(y) = \{\phi(t, y) \mid 0 \le t < \infty\}$

ϕ is solution of $\dot{x} = f(x)$ starting at $\phi(0, y) = y$

-ve semi orbit through $y \Rightarrow \gamma^-(y) = \{\phi(t, y) \mid -\infty < t \le 0\}$

p is a +ve limit point of solution $\phi(t, y)$ (for $t \ge 0$) if there is a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ s.t $\phi(t_n, y) \to p$ as $n \to \infty$

The set of all +ve limit points of $\phi(t,y)$ is L^+, the +ve limit set of $\phi(t,y)$.

If $\phi(t,y)$ is bounded, then its +ve limit set is non empty, invariant, compact, and $\phi(t,y)$ approaches its positive limit set as $t \to \infty$ (see B.1)

(Poincare – Bendixson)

Let γ^+ be a bounded +ve semi orbit of $\dot{x} = f(x)$ and L^+ be its +ve limit set. If L^+ contains no equilibrium points, then it is a periodic orbit.

(B.10)

(Bendixson Criterion)

If on a simply connected region $D \subset R^2$ the expression $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is not identically zero and does not change sign, then $\dot{x} = f(x)$ has no periodic orbits lying entirely in D.

ex.
$\dot{x}_1 = x_2 + x_1 x_2^2$
$\dot{x}_2 = -x_1 + x_1^2 x_2$
$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = x_1^2 + x_2^2$
never changes sign
$\therefore$ no limit cycles.

$\frac{dx_2}{dx_1} = \frac{f_2}{f_1} \Rightarrow f_2 dx_1 - f_1 dx_2 = 0$ (true for any trajectory including limit cycle)

For a limit cycle

$$\int_C (f_2 dx_1 - f_1 dx_1) = 0 \Rightarrow$$ using Stoke's theorem $$\iint \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) dx_1 dx_2 = 0$$

QED.

(Poincare)

If a limit cycle exists in a second order system then $N = S+1$, where N = # of nodes, centers and foci enclosed by a limit cycle and S is the # of enclosed saddle points.

Oscillation

when

$$x(t+T) = x(t)$$

with a non trivial solution

i.e $x(t) \neq 0$

For linear systems

when $\lambda = \pm j\beta$, we see oscillations (harmonic oscillator)

① unstable (any perturbation can destabilize it)

② amplitude dependent on initial conditions.

non linear oscillations (limit cycles)

① can be stable

② amplitude independent of initial conditions

Numerical Construction of Phase Portraits

① Select areas around equilibrium points

② choose initial points

③ Take each initial point x_0 and solve for trajectories in forward time by $\dot{x} = f(x)$, $x(0) = x_0$ and backward time by

$$\dot{x} = -f(x), \quad x(0) = x_0$$

Homework: P_{2n} 1.15, 1.16, 1.17, 1.18

~~draw~~ PS# analyse limit cycle for

$$\dot{x}_1 = x_2 + x_1(x_1^2 + x_2^2 - 1)$$

$$\dot{x}_2 = -x_1 + x_2(x_1^2 + x_2^2 - 1)$$