INPUT-OUTPUT LINEARIZATION OF SISO Systems

Example ①

$$\dot{x}_1 = a\sin x_2$$
$$\dot{x}_2 = -x_1^2 + u$$
$$y = x_1$$
$$\dot{y} = \dot{x}_1 = a\sin x_2$$
$$\ddot{y} = \ddot{x}_1 = a\cos x_2 \dot{x}_2 = -a x_1^2 \cos x_2 + a\cos x_2 u$$
$$\boxed{u = -(a\cos x_2)^{-1}\left[a x_1^2 \cos x_2 + v\right]}$$
$$\Rightarrow \ddot{y} = v \quad , \text{ Take } v = \ddot{y}_d - k_1(\dot{y}-\dot{y}_d) - k_2(y-y_d)$$
$$(\ddot{y}-\ddot{y}_d) + k_1(\dot{y}-\dot{y}_d) + k_2(y-y_d) = 0$$
$$\Rightarrow y \to y_d$$
If $y_d \equiv 0$, then $y \to 0$

Example ②

$$\dot{x}_1 = a\sin x_2$$
$$\dot{x}_2 = -x_1^2 + u$$
$$y = x_2$$
$$\dot{y} = \dot{x}_2 = -x_1^2 + u$$
Take $\boxed{u = x_1^2 + \dot{y}_d - k(y-y_d)}$
$$\Rightarrow \dot{y} = \dot{y}_d - k(y-y_d) \Rightarrow (y-y_d) \to 0$$
if $y_d \equiv 0 \Rightarrow y \to 0$

Notice x_1 dynamics (e.g for regulation control, i.e $y_d \equiv 0$), then $y \to 0$ (i.e $x_2 \to 0$) independent of x_1

then $a\sin x_2 \to 0$

i.e $\dot{x}_1 \to 0$

or precisely $x_2 = x_2(0)\, e^{-kt}$

$$\therefore \dot{x}_1 = a\sin(x_2(0)\, e^{-kt})$$
$$x_1 = \int_0^t a\sin(x_2(0)\, e^{-k\tau})\, d\tau$$

example ③

$$\dot{x}_1 = 2x_1$$
$$\dot{x}_2 = -x_1^2 + u$$
$$y = x_2$$
from example ②, $u = x_1^2 - ky$, s.t $y \to 0$

but $x_1 \to \infty$

example ④ $\dot{x}_1 = 2x_1$, $\dot{x}_2 = -x_1^2 + u$ $\qquad y = x_1$, $\dot{y} = \dot{x}_1 = 2x_1$, $\ddot{y} = 2\dot{x}_1 = 4x_1$ ... etc.

System

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

$x \in R^n$, u and y scalars

$$\left.\begin{aligned} \dot{x}_1 &= \sin x_2 + (x_2+1)x_3 \\ \dot{x}_2 &= x_1^5 + x_3 \\ \dot{x}_3 &= x_1^2 + u \end{aligned}\right\} n=3$$

$$y = x_1$$

$$\dot{y} = \dot{x}_1 = \sin x_2 + (x_2+1)x_3$$

$$\ddot{y} = f_1(x) + (x_2+1)u \quad \text{— ①}$$

where $f_1(x) = (x_1^5 + x_3)(x_3 + \cos x_2) + (x_2+1)x_1^2$

$$\boxed{u = \frac{1}{x_2+1}(v - f_1)}$$

$$\Rightarrow \ddot{y} = v$$

$$v = \ddot{y}_d - k_2(\dot{y} - \dot{y}_d) - k_1(y - y_d)$$

$$\Rightarrow (y - y_d) \to 0$$

- control law defined everywhere except at $x_2 = -1$
- full state feedback
- relative degree $= 2$ (2 differentiations) rel. degree $= r$
  (If you differentiate y n times and still don't get u, then system is not controllable)

$\because$ rel. degree $< n$ $\therefore$ one state is unobservable by ①

we can use $e, \dot{e}$ and x_3 as the three states $(e = y - y_d)$

internal dynamics (dynamics of the unobservable state)

$$\dot{x}_2 = x_1^2 + \frac{1}{x_2+1}(\ddot{y}_d - k_1 e - k_2 \dot{e} + f_1)$$

we want stable internal dynamics

Example $\quad \dot{x}_1 = x_2^3 + u \quad -①$

$\dot{x}_2 = u \quad -②$

$y = x_1$

$\dot{y} = \dot{x}_1 = x_2^3 + u$

$\therefore \boxed{u = -x_2^3 - e(t) + \dot{y}_d(t)} \quad -③$ where $e(t) = y(t) - y_d(t)$

makes $x_1 \to 0$ by obtaining $\dot{e} + e = 0$

However, dynamics of x_2 are : (using ③ in ②)

$$\dot{x}_2 = -x_2^3 - e(t) + \dot{y}_d(t)$$

Since $e(t) \to 0$ exponentially and $\dot{y}_d$ is bounded, we have

$|\dot{y}_d(t) - e(t)| < d$ (a positive number)

$\therefore \dot{x}_2 + x_2^3 = \dot{y}_d - e \Rightarrow$

If $x_2 > d^{1/3}$, then $\dot{x}_2 < 0$

If $x_2 < -d^{1/3}$ then $\dot{x}_2 > 0$

$\therefore |x_2(t)| \le d^{1/3}$

$\therefore$ Internal dynamics keep $x_2(t)$ bounded.

In general, difficult to analyze internal dynamics.

Look at $\left.\begin{aligned} \dot{x}_1 &= x_2 + u \\ \dot{x}_2 &= u \\ y &= x_1 \end{aligned}\right\}$ $\dfrac{Y(s)}{U(s)} = \dfrac{s+1}{s^2}$

stable internal dynamics.

look at $\left.\begin{aligned} \dot{x}_1 &= x_2 + u \\ \dot{x}_2 &= -u \\ y &= x_1 \end{aligned}\right\}$ $\dfrac{Y(s)}{U(s)} = \dfrac{s-1}{s^2}$

unstable internal dynamics

$\therefore$ for linear systems internal dynamics related to zeroes.

Zero Dynamics: is the internal dynamics of the system when the system output is kept at zero by the input.

e.g for $\dot{x}_1 = x_2^3 + u$

$\dot{x}_2 = u$

$y = x_1$

$\therefore$ here $u = -x_2^3$

Internal Dynamics become

$\dot{x}_2 + x_2^3 = 0$

(stable using $V = x_2^2$)

Result: For nonlinear systems for tracking problem, local exponential stability of the zero dynamics guarantees stable internal dynamics if $y_d, \dots, y_d^{(r-1)}$ have small magnitudes.

For regulation problems, local asymptotic stability of zero dynamics $\Rightarrow$ local asymptotic stability of internal dynamics.

INPUT / OUTPUT LINEARIZATION

$\dot{x} = f + g u$ (nth order)

$y = h(x)$

$\dot{y} = \nabla h \cdot \dot{x} = \nabla h (f + gu) = L_f h(x) + L_g h(x) u$

If $L_g h(x) \neq 0$, then $u = \frac{1}{L_g h}(-L_f h + v)$ gives control,

otherwise

$\ddot{y} = L_f^2 h(x) + L_g L_f h(x) u$

If $L_g L_f h(x)$ is again zero, then keep differentiating till we find r ($< n$) s.t. $\quad r$: relative degree

$L_g L_f^{r-1} h(x) \neq 0$

then take $u = \frac{1}{L_g L_f^{r-1} h}(-L_f^r h + v)$

giving $y^{(r)} = v$

NORMAL FORM.

$\dot{x} = f + gu$

$y = h(x)$

with relative degree $r < n$ can be written as

$$\frac{d}{dt}\begin{bmatrix} \xi_1 \\ \vdots \\ \xi_{r-1} \\ \xi_r \end{bmatrix} = \begin{bmatrix} \xi_2 \\ \vdots \\ \xi_r \\ a(\xi,\eta) + b(\xi,\eta)u \end{bmatrix}$$

$\dot{\eta} = w(\xi,\eta)$

$y = \xi_1$

where $\xi = [h \quad L_f h \ \ldots \ L_f^{r-1} h]^T$

$\eta = [\eta_1 \ \eta_2 \ldots \eta_{n-r}]^T$

example

$$\dot{x} = \begin{pmatrix} -x_1 \\ x_1 x_2 \\ x_2 \end{pmatrix} + \begin{pmatrix} e^{x_2} \\ 1 \\ 0 \end{pmatrix} u$$

$y = h(x) = x_3$

$\dot{y} = x_2$

$\ddot{y} = \dot{x}_2 = x_1 x_2 + u$

$\therefore \xi_1 = x_3 = h(x)$

$\xi_2 = L_f h(x) = x_2$

$\eta(x)$ should satisfy

$$L_g \eta = \frac{\partial \eta}{\partial x_1} e^{x_2} + \frac{\partial \eta}{\partial x_2} = 0$$

e.g. $\eta(x) = 1 + x_1 - e^{x_2}$

$\therefore z = (\xi_1, \xi_2, \eta)^T$

MIMO Systems. $\dot{y}_j = L_f h_j + \sum_{i=1}^{m} (L_{g_i} h_j) u_i$ (take each output & differentiate successively to get input)

we can get

$$\begin{bmatrix} y_1^{(r_1)} \\ \vdots \\ y_m^{(r_m)} \end{bmatrix} = \begin{bmatrix} L_f^{r_1} h_1(x) \\ \vdots \\ L_f^{r_m} h_m(x) \end{bmatrix} + E(x) \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

decoupling matrix